МАТЕМАТИКА

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ON SOLUTIONS OF INTEGRAL EQUATIONS WITH REFLECTIONS

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In this paper, we deal with some classes of singular integral equations on the real axes with reflections of the form

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau = f(t) \qquad (01)$$

and

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau+\sum_{j=1}^{m}\int_{\mathbb{R}}a_j(t)b_j(\tau)\varphi(\tau)d\tau = f(t). \qquad (02)$$

By means of the Riemann boundary value problems and of the systems of linear algebraic equations, we give an algebraic method to obtain all solutions of equations (01) and (02) in a closed form. Note that some special cases of normal sovability of (01) have been considered in [2 – 3].

1. Introduction. Let $X = H^{\mu}(\mathbb{R}), (0<\mu\le 1)$ be the Holder space on $\mathbb{R}$. Consider the following operators in *X*:

$$(l\varphi)(t) = \int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau, \qquad (1.1)$$

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\mathbb{R}}\frac{\varphi(\tau)d\tau}{\tau-t}, \qquad (1.2)$$

where $l(\tau,t)$ is a given function satisfying the Holder condition in $(\tau,t)\in\mathbb{R}\times\mathbb{R}$.

Definition 1.1 (see [2; 4]). *We say that the function* $l(\tau,t)$ *belongs to* $H_p^{++}(1<p<\infty)$ *if:*

(a) $l(z,\zeta)$ *is analytic in* z *and* ζ *is in the upper half-plane* $\mathbb{C}^+$ *(if one variable is fixed, then* $l(\tau,t)$ *is analytic in* $\mathbb{C}^+\cup\mathbb{R}$*);*

(b) $\int_{\mathbb{R}}|l(\tau+iy,x)|^r\,d\tau\le const,\ r>1$ *for almost* $x\in\mathbb{R}$, *where constant is independent of* y, $y\ge 0$;

(c) $\|l_y\|_{L_p\to L_p}<const$, *where*

$$(l_y\varphi)(x+iy) = \int_{\mathbb{R}}l(\tau,x+iy)\varphi(\tau)d\tau.$$

Write

$$(W\varphi)(t)=\varphi(-t),\ \ Q_1=\frac{1}{2}(I+W),\ \ Q_2=\frac{1}{2}(I-W); \qquad (1.3)$$

$$P_1=\frac{1}{2}(I+S),\ \ P_2=\frac{1}{2}(I-S). \qquad (1.4)$$

It is easy to check that (see [2])

$$SW=-WS,\ \ SQ_i=Q_jS,\ \ WS_i=P_jW,\ \ i\ne j,\ \ i,j=1,2;$$

$$X=X_1\oplus X_2=X^+\oplus X^-,\ X_j=Q_jX,\ X^+=P_1X,\ X^-=P_2X. \qquad (1.5)$$

We consider the solvability of the singular integral equations (in *X*) of the following form

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau = f(t) \qquad (1.6)$$

and

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}} l(\tau,t)\varphi(\tau)d\tau+\sum_{j=1}^{m}\int_{\mathbb{R}} a_j(t)b_j(\tau)\varphi(\tau)d\tau = f(t), \qquad (1.7)$$

where $a_1, a_2, b_+, a_j, b_j \in X$ $(j=1,2,\ldots,m)$ are given.

2. The solvability of equation (1.6)

Rewrite the equation (1.6) in the form

$$a_+(t)(Q_1\varphi)(t)+a_-(t)(Q_2\varphi)(t)+b_+(t)(SQ_1\varphi)(t)+(l\varphi)(t)=f(t), \qquad (2.1)$$

where Q_1, Q_2, S, l are the operators defined by (1.1) – (1.3) and $a_\pm(t)=a_1(t)+a_2(t)$. In this paper, we shall assume that

$$l(-\tau,t)=l(\tau,t),\ t\in\mathbb{R}. \qquad (2.2)$$

It is easy to see that the equation (2.1) is equivalent to the system:

$$\begin{cases} A_1(t)(Q_1\varphi)(t)+C_2(t)(Q_2\varphi)(t)+B_2(t)(SQ_1\varphi)(t)+(Q_1lQ_1\varphi)(t)=f_1(t),\\ A_2(t)(Q_1\varphi)(t)+C_1(t)(Q_2\varphi)(t)+B_1(t)(SQ_1\varphi)(t)+(Q_2lQ_1\varphi)(t)=f_2(t),\end{cases}$$

and this is a consequence of the assumption (2.2), where

$$A_{1,2}(t)=\frac{1}{2}(a_+(t)\pm a_+(-t)),\ B_{1,2}(t)=\frac{1}{2}(b_+(t)\pm b_+(-t)),$$

$$C_{1,2}(t)=\frac{1}{2}(a_-(t)\pm a_-(-t)),\ f_{1,2}(t)=\frac{1}{2}(f(t)\pm f(-t)).$$

Write $\varphi_1(t)=(Q_1\varphi)(t)$ and $\varphi_2(t)=(Q_2\varphi)(t)$, then $\varphi_j\in X_j$ for $j=1,2$. Hence, we get the following system in $X_1\times X_2$:

$$\begin{cases} A_1(t)\varphi_1(t)+C_2(t)\varphi_2(t)+B_2(t)(S\varphi_1)(t)+(Q_1l\varphi_1)(t)=f_1(t),\\ A_2(t)\varphi_1(t)+C_1(t)\varphi_2(t)+B_1(t)(S\varphi_1)(t)+(Q_2l\varphi_1)(t)=f_2(t).\end{cases} \qquad (2.3)$$

Lemma 2.1. *If* (φ_1,φ_2) *is a solution of the equation* (2.3) *in* $X\times X$, *then* $(Q_1\varphi_1, Q_2\varphi_2)$ *is its solution in* $X_1\times X_2$.

Proof. Using the representation $\varphi_j=Q_1\varphi_j+Q_2\varphi_j$ we can write (2.3) in the form:

$$\begin{aligned}&A_1(t)(Q_1\varphi_1)(t)+C_2(t)(Q_2\varphi_2)(t)+B_2(t)(SQ_1\varphi_1)(t)+(Q_1lQ_1\varphi_1)(t)-f_1(t)=\\&=-[A_1(t)(Q_2\varphi_1)(t)+C_2(t)(Q_1\varphi_2)(t)+B_2(t)(SQ_2\varphi_1)(t)]-\\&-[A_2(t)(Q_2\varphi_1)(t)+C_1(t)(Q_1\varphi_2)(t)+B_1(t)(SQ_2\varphi_1)(t)]=\\&=A_2(t)(Q_1\varphi_1)(t)+C_1(t)(Q_2\varphi_2)(t)+B_1(t)(SQ_1\varphi_1)(t)+(Q_2lQ_1\varphi_1)(t)-f_1(t).\end{aligned}$$

Note that

$$A_1(t)(Q_1\varphi_1)(t)=\frac{1}{2}Q_1(a_+(t)\varphi_1(t)+a_+(t)\varphi_1(-t))\in X_1,$$

$$C_2(t)(Q_2\varphi_2)(t)=\frac{1}{2}Q_1(a_-(t)\varphi_2(t)-a_-(t)\varphi_2(-t))\in X_1,$$

$$B_2(t)(SQ_1\varphi_1)(t)=\frac{1}{2}Q_1(b_+(t)(S\varphi_1)(t)-b_+(-t)(S\varphi_1)(t))\in X_1,$$

$$(Q_1lQ_1\varphi_1)(t)\in X_1,\ f_1(t)=(Q_1f)(t)\in X_1.$$

$$A_1(t)(Q_2\varphi_1)(t)=\frac{1}{2}Q_2(a_+(t)\varphi_1(t)+a_+(-t)\varphi_1(t))\in X_2,$$

$$C_2(t)(Q_1\varphi_2)(t)=\frac{1}{2}Q_2(a_-(t)\varphi_2(t)+a_-(t)\varphi_2(-t))\in X_2,$$

$$B_2(t)(SQ_2\varphi_1)(t)=\frac{1}{2}Q_2(b_+(t)(S\varphi_1)(t)-b_+(-t)(S\varphi_1)(t))\in X_2.$$

Similarly, it is easy to see that all the left sides of this system belong to X_1; however, the right sides belong to X_2. From (1.5), both sides are equal to zero, which was to be proved.

Thus, it is enough to consider the system (2.3) in the space $X \times X$ only.

From system (2.3), we have:

$$C_1(t) f_1(t) - C_2(t) f_2(t) = [A_1(t)C_1(t) - A_2(t)C_2(t)]\varphi_1(t) +$$
$$+[C_1(t)B_2(t) - C_2(t)B_1(t)](S\varphi_1)(t) + C_1(t)(Q_1 l\varphi_1)(t) - C_2(t)(Q_2 l\varphi_1)(t),$$

$$A_1(t) f_2(t) - A_2(t) f_1(t) = [A_1(t)C_1(t) - A_2(t)C_2(t)]\varphi_2(t) +$$
$$+[A_1(t)B_1(t) - A_2(t)B_2(t)](S\varphi_1)(t) + A_1(t)(Q_2 l\varphi_1)(t) - A_2(t)(Q_1 l\varphi_1)(t).$$

2.1. Case of $A_1(t)C_1(t) - A_2(t)C_2(t) \neq 0, \forall t \in$ □

Now we consider the case of $A_1(t)C_1(t) - A_2(t)C_2(t) \neq 0, \forall t \in$ □. Then (2.3) can be rewritten in the form

$$\begin{cases} u(t)\varphi_1(t) + v(t)(S\varphi_1)(t) + Q_1[a_-(-t)(l\varphi_1)(t)] = C_1(t) f_1(t) - C_2(t) f_2(t), \\ u(t)\varphi_2(t) + v_1(t)(S\varphi_1)(t) + Q_2[a_+(-t)(l\varphi_1)(t)] = A_1(t) f_2(t) - A_2(t) f_1(t), \end{cases} \tag{2.4}$$

where

$$u(t) = \frac{1}{2}[a_+(-t)a_-(t) + a_+(t)a_-(-t)] = Q_1[a_+(t)a_-(-t)],$$
$$v(t) = \frac{1}{2}[b_+(t)a_-(-t) - a_-(t)b_+(-t)] = Q_2[b_+(t)a_-(-t)],$$
$$v_1(t) = \frac{1}{2}[a_+(t)b_+(-t) + a_+(-t)b_+(t)] = Q_1[a_+(t)b_+(-t)].$$

THEOREM 2.1. *Suppose that the function* $l(\tau,t)$ *satisfies the condition* (2.2), *i.e.* $l(-\tau,t) = l(\tau,t)$ *and*

$$[u(t) + v(t)]^{-1} Q_1[a_-(-t)(l\varphi_1)(t)] \in H_p^{++}, \tag{2.5}$$

then the equation (2.5) admits all solutions in a closed form

$$\varphi(t) = (Q_1\varphi_1)(t) + (Q_2\varphi_2)(t),$$

where $(\varphi_1(t), \varphi_2(t))$ *is a solution of the system* (2.4) *in* $X \times X$.

Proof. Put

$$\Phi_1(z) = \frac{1}{2\pi i} \int_{□} \frac{\varphi_1(\tau)}{\tau - z} d\tau.$$

According to Sokhotski – Plemelij formula, we have

$$\begin{cases} \varphi_1(t) = \Phi_1^+(t) - \Phi_1^-(t) \\ (S\varphi_1)(t) = \Phi_1^+(t) + \Phi_1^-(t). \end{cases} \tag{2.6}$$

The first equation of system (2.4) can be written in the form (2.7)

$$u(t)[\Phi_1^+(t) - \Phi_1^-(t)] + v(t)[\Phi_1^+(t) + \Phi_1^-(t)] + Q_1[a_-(-t)l(\Phi_1^+(t) - \Phi_1^-(t))] = C_1(t) f_1(t) - C_2(t) f_2(t).$$

By [1] (Lemma 5.1), we obtain $l\Phi_1^+(t) = 0, l\Phi_1^-(t) \in X^+$ and

$$\Phi_1^+(t) - \frac{Q_1[a_-(-t)l\Phi_1^-(t)]}{u(t)+v(t)} = \frac{u(t)-v(t)}{u(t)+v(t)}\Phi_1^-(t) + \frac{C_1(t) f_1(t) - C_2(t) f_2(t)}{u(t)+v(t)}. \tag{2.7}$$

Put

$$\begin{cases} \Phi^+(t) = \Phi_1^+(t) - \dfrac{Q_1[a_-(-t)l\Phi_1^-(t)]}{u(t)+v(t)} \in X^+, \\ \Phi^-(t) = \Phi_1^-(t), \end{cases} \tag{2.8}$$

we reduce the first equation of system (2.4) to the following Riemann boundary problem

$$\Phi^{+}(t) = G(t)\Phi^{-}(t) + g(t), \tag{2.9}$$

where $G(t) = \dfrac{u(t) - v(t)}{u(t) + v(t)}$, $g(t) = \dfrac{C_1(t) f_1(t) - C_2(t) f_2(t)}{u(t) + v(t)}$.

Suppose that $u^2(t) - v^2(t)$ is a non-vanishing function on $\mathbb{R}$. Then $G(t), g(t) \in X$ and $G(t) \neq 0$ for any $t \in \mathbb{R}$. Put

$$\varkappa = Ind G(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} d\ln G(t),$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \ln\left[\left(\frac{\tau - i}{\tau + i}\right)^{-\varkappa} G(\tau)\right] \cdot \frac{d\tau}{\tau - z},$$

$$X^{+}(z) = e^{\Gamma^{+}(z)},\ X^{-}(z) = \left(\frac{z - i}{z + i}\right)^{-\varkappa} e^{\Gamma^{-}(z)}.$$

Using the results of Riemann boundary problem, we have to consider the following cases:

1. If $\varkappa \geq 0$ then the problem (2.9) is solvable and has the general solution given by formula

$$\Phi(z) = X(z)\left[\Psi(z) + \frac{P_{\varkappa-1}(z)}{(z + i)^{\varkappa}}\right], \tag{2.10}$$

where

$$\Psi(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{g(\tau)}{X^{+}(\tau)} \frac{d\tau}{\tau - z} \tag{2.11}$$

and $P_{\varkappa-1}(z) = p_1 + p_2 z + \cdots + p_{\varkappa} z^{\varkappa-1}$ is a polynomial of degree $\varkappa - 1$ with arbitrary complex coefficients.

2. If $\varkappa < 0$ then the necessary condition for the problem (2.9) to be solvable is that

$$\int_{-\infty}^{+\infty} \frac{g(\tau)}{X^{+}(\tau)} \frac{d\tau}{(\tau + i)^{k}} = 0, k = 1, 2, \ldots, -\varkappa .$$

Last conditions can be written as follows

$$\int_{-\infty}^{+\infty} \frac{C_1(\tau) f_1(\tau) - C_2(\tau) f_2(\tau)}{X^{+}(\tau)(u(\tau) + v(\tau))} \frac{d\tau}{(\tau + i)^{k}} = 0, k = 1, 2, \ldots, -\varkappa . \tag{2.12}$$

If the condition (2.12) is satisfied then all solutions are given by formula

$$\Phi(z) = X(z)\Psi(z).$$

Hence, we have

$$\varphi_1(t) = \Phi_1^{+}(t) - \Phi_1^{-}(t) = \Phi^{+}(t) + \frac{Q_1[a_{-}(-t) l \Phi^{-}(t)]}{u(t) + v(t)} - \Phi^{-}(t)$$

and $\varphi_2(t)$ is defined by (2.4). The proof is complete.

2.2. Case of $A_1(t)C_1(t) - A_2(t)C_2(t) \equiv 0.$

THEOREM 2.2. *Suppose that the function* $l(\tau, t)$ *satisfies the condition* (2.2), *i.e.* $l(-\tau, t) = l(\tau, t)$. *Consider the case* $u(t) \equiv 0$ *and*

$$\begin{cases} Q_2[b_{+}(t) a_{-}(-t)] \equiv 0, \\ Q_1[a_{+}(t) b_{+}(-t)] \equiv 0. \end{cases}$$

If it is the case, the equation (2.1) *admits all solutions in a closed form*

$$\varphi(t) = (Q_1\varphi_1)(t) + (Q_2\varphi_2)(t),$$

where $(\varphi_1(t), \varphi_2(t))$ *is a solution of the system* (2.4) in $X \times X$.

Proof. Note that

$$C_1(t)f_1(t)-C_2(t)f_2(t)=Q_1[a_-(t)f(-t)],$$
$$A_1(t)f_2(t)-A_2(t)f_1(t)=Q_2[a_+(-t)f(t)]$$

and (2.4) is equivalent to the system

$$\begin{cases} Q_1[a_+(t)a_-(-t)]\varphi_1(t)+Q_2[b_+(t)a_-(-t)](S\varphi_1)(t)+Q_1[a_-(-t)(l\varphi_1)(t)]=Q_1[a_-(t)f(-t)],\\ Q_1[a_+(t)a_-(-t)]\varphi_2(t)+Q_1[a_+(t)b_+(-t)](S\varphi_1)(t)+Q_2[a_+(-t)(l\varphi_1)(t)]=Q_2[a_+(-t)f(t)]. \end{cases} \quad (2.13)$$

The first equation of system (2.13) can be rewritten in the form

$$\begin{aligned} &Q_1[a_+(t)a_-(-t)](Q_1\varphi_1)(t)\quad +Q_2[b_+(t)a_-(-t)](Q_2S\varphi_1)(t)+\\ &+Q_1[a_-(-t)(l\varphi_1)(t)]=Q_1[a_-(t)f(-t)]. \end{aligned} \quad (2.14)$$

$$Q_1[a_+(t)a_-(-t)](Q_2\varphi_1)(t)+Q_2[b_+(t)a_-(-t)](Q_1S\varphi_1)(t)=0. \quad (2.15)$$

Rewrite the second equation of system (2.13) in the form

$$\begin{aligned} &Q_1[a_+(t)a_-(-t)](Q_2\varphi_2)(t)+Q_1[a_+(t)b_+(-t)](Q_2S\varphi_1)(t)+\\ &+Q_2[a_+(-t)(l\varphi_1)(t)]=Q_2[a_+(-t)f(t)]. \end{aligned} \quad (2.16)$$

$$Q_1[a_+(t)a_-(-t)](Q_1\varphi_2)(t)+Q_1[a_+(t)b_+(-t)](Q_1S\varphi_1)(t)=0. \quad (2.17)$$

Note that

$$Q_1[a_+(t)a_-(-t)](Q_1\varphi_1)(t)=Q_1[a_+(t)a_-(-t)(Q_1\varphi_1)(t)]\in X_1,$$

$$Q_2[b_+(t)a_-(-t)](Q_2S\varphi_1)(t)=Q_2[b_+(t)a_-(-t)(Q_2S\varphi_1)(t)]\in X_2,$$

$$Q_1[a_+(t)a_-(-t)](Q_2\varphi_2)(t)=Q_2[a_+(t)a_-(-t)(Q_2\varphi_2)(t)]\in X_2$$

and

$$Q_1[a_+(t)b_+(-t)](Q_2S\varphi_1)(t)=Q_1[a_+(t)b_+(-t)(Q_2S\varphi_1)(t)]\in X_1.$$

Hence, equation (2.14) and (2.16) are equivalent to the systems:

$$\begin{cases} Q_1[a_+(t)a_-(-t)(Q_1\varphi_1)(t)]+Q_1[a_-(-t)(l\varphi_1)(t)]=Q_1[a_-(t)f(-t)],\\ Q_2[b_+(t)a_-(-t)](Q_2S\varphi_1)(t)=0; \end{cases}$$

$$\begin{cases} Q_2[a_+(t)a_-(-t)(Q_2\varphi_2)(t)]+Q_2[a_+(-t)(l\varphi_1)(t)]=Q_2[a_+(-t)f(t)],\\ Q_1[a_+(t)b_+(-t)](Q_2S\varphi_1)(t)=0. \end{cases}$$

Under the assumption $u(t)\equiv 0$ i.e. $Q_1[a_+(t)a_-(-t)]\equiv 0.$ If it is the case, equation (2.15) and equation (2.17) are equivalent to the system:

$$\begin{cases} Q_2[b_+(t)a_-(-t)](Q_1S\varphi_1)(t)\equiv 0,\\ Q_1[a_+(t)b_+(-t)](Q_1S\varphi_1)(t)\equiv 0. \end{cases}$$

Since $Q_2[b_+(t)a_-(-t)]\equiv 0$ and $Q_1[a_+(t)b_+(-t)]\equiv 0,$ then equation (2.13) is equivalent to the system:

$$\begin{cases} a_+(t)a_-(-t)(Q_1\varphi_1)(t)+a_-(-t)(l\varphi_1)(t)=a_-(t)f(-t)+z_2(t),\\ a_+(t)a_-(-t)(Q_2\varphi_2)(t)+a_+(-t)(l\varphi_1)(t)=a_+(-t)f(t)+z_1(t), \end{cases} \quad (2.18)$$

where $z_1(t)$ is an arbitrary function in X_1, $z_2(t)$ is an arbitrary function in X_2.

a. Suppose that $a_+(t)a_-(-t)\neq 0, \forall t\in$.

Since $lQ_2\varphi_1=0,$ system (2.18) can be written in the form

$$\begin{cases} a_+(t)a_-(-t)(Q_1\varphi_1)(t)+a_-(-t)(lQ_1\varphi_1)(t)=a_-(t)f(-t)+z_2(t),\\ a_+(t)a_-(-t)(Q_2\varphi_2)(t)+a_+(-t)(lQ_1\varphi_1)(t)=a_+(-t)f(t)+z_1(t). \end{cases} \quad (2.19)$$

Since $a_+(t)a_-(-t) \neq 0, \forall t \in \mathbb{R}$, the first equation of the system (2.19) is solvable in a closed form, and $(Q_2\varphi_2)(t)$ is defined by the second equation of (2.19).

If it is the case, the solutions of (2.1) are of the form

$$\varphi(t) = (Q_1\varphi_1)(t) + (Q_2\varphi_2)(t),$$

where $(Q_1\varphi_1)(t), (Q_2\varphi_2)(t)$ is the solution of the system (2.19).

b. If $a_+(t_0) = 0, a_-(-t_0) \neq 0,$ the necessary condition for the system (2.18) to be solvable is that

$$a_-(-t_0)(l\varphi_1)(t_0) = a_-(t_0) f(-t_0) + z_2(t_0)$$

$$a_+(-t_0)(l\varphi_1)(t_0) = a_+(-t_0) f(t_0) + z_1(t_0).$$

It follows

$$(l\varphi_1)(t_0) = \frac{a_-(t_0) f(-t_0) + z_2(t_0)}{a_-(-t_0)} = \frac{a_+(-t_0) f(t_0) + z_1(t_0)}{a_+(-t_0)}. \qquad (2.20)$$

If the condition (2.20) is satisfied, since the solution belongs to Holder space $H^\mu(\mathbb{R})$, implies (2.18) has solution if $\left|M(t) - M(t_0)\right| = o(|t - t_0|^\mu),\ \lim_{t \to t_0} M(t) = c_1 \in (-\infty, +\infty),$ where

$$M(t) = \frac{a_-(t) f(-t) + z_2(t) - a_-(-t)(l\varphi_1)(t)}{a_+(t)a_-(-t)}.$$

$\left|W(t) - W(t_0)\right| = o(|t - t_0|^\mu),\ \lim_{t \to t_0} W(t) = c_2 \in (-\infty, +\infty),$ where

$$W(t) = \frac{a_+(-t) f(t) + z_1(t) - a_+(-t)(l\varphi_1)(t)}{a_+(t)a_-(-t)}.$$

If it is the case, then $(Q_1\varphi_1)(t_0) = c_1, (Q_2\varphi_2)(t_0) = c_2.$

c. If $a_-(-t_0) = 0,$ the necessary condition for the system (2.18) to be solvable is that:

$$a_-(t_0) f(-t_0) + z_2(t_0) = 0,$$

$$a_+(-t_0)(l\varphi_1)(t_0) = a_+(-t_0) f(t_0) + z_1(t_0).$$

Since the solution belongs to Holder space $H^\mu(\mathbb{R})$, implies (2.18) has solution if

$$\left| \frac{a_-(t) f(-t) + z_2(t)}{a_+(t)a_-(-t)} - \frac{a_-(t_0) f(-t_0) + z_2(t_0)}{a_+(t_0)a_-(-t_0)} \right| = o(|t - t_0|^\mu),$$

$$\lim_{t \to t_0} \frac{a_-(t) f(-t) + z_2(t)}{a_+(t)a_-(-t)} = d_1 \in (-\infty, +\infty),$$

$\left|T(t) - T(t_0)\right| = o(|t - t_0|^\mu),\ \lim_{t \to t_0} T(t) = d_2 \in (-\infty, +\infty),$ where

$$T(t) = \frac{a_+(-t) f(t) + z_1(t) - a_+(-t)(l\varphi_1)(t)}{a_+(t)a_-(-t)}.$$

If it is the case, then $(Q_1\varphi_1)(t_0) = c_1, (Q_2\varphi_2)(t_0) = c_2$ and the solutions $\varphi(t_0)$ of (2.1) are written in the form

$$\varphi(t_0) = (Q_1\varphi_1)(t_0) + (Q_2\varphi_2)(t_0).$$

3. The solvability of equation (1.7)

We consider the solvability of the singular integral equations (in X) of the following form

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau+\sum_{j=1}^{m}\int_{\mathbb{R}}a_j(t)b_j(\tau)\varphi(\tau)d\tau=f(t). \tag{3.1}$$

Denote by N_{b_j}, $j=1,\dots,m,$ the linear functionals on X defined as follows

$$(N_{b_j}\varphi)=\int_{\mathbb{R}}b_j(\tau)\varphi(\tau)d\tau,\ \varphi\in X.$$

Put $(N_{b_j}\varphi)=\lambda_j$, $j=1,2,\dots,m.$ We reduce equation (3.1) to the following problem: find solutions φ of equation

$$a_1(t)\varphi(t)+a_2(t)\varphi(-t)+\frac{b_+(t)}{\pi i}\int_{\mathbb{R}}\frac{t\varphi(\tau)d\tau}{\tau^2-t^2}+\int_{\mathbb{R}}l(\tau,t)\varphi(\tau)d\tau=f(t)-\sum_{j=1}^{m}\lambda_j a_j(t) \tag{3.2}$$

depended on the parameters $\lambda_1,\dots,\lambda_m$ and fulfilled the following conditions

$$(N_{b_j}\varphi)=\lambda_j,\ j=1,2,\dots,m. \tag{3.3}$$

Rewrite this equation in the form

$$a_+(t)(Q_1\varphi)(t)+a_-(t)(Q_2\varphi)(t)+b_+(t)(SQ_1\varphi)(t)+(l\varphi)(t)=f(t)-\sum_{j=1}^{m}\lambda_j a_j(t), \tag{3.4}$$

where $a_\pm(t)=a_1(t)+a_2(t).$ In the sequel. we shall assume that

$$l(-\tau,t)=l(\tau,t). \tag{3.5}$$

The equation (3.2) is equivalent to the symtem:

$$\begin{cases}A_1(t)(Q_1\varphi)(t)+C_2(t)(Q_2\varphi)(t)+B_2(t)(SQ_1\varphi)(t)+(Q_1lQ_1\varphi)(t)=f_1^*(t),\\ A_2(t)(Q_1\varphi)(t)+C_1(t)(Q_2\varphi)(t)+B_1(t)(SQ_1\varphi)(t)+(Q_2lQ_1\varphi)(t)=f_2^*(t),\end{cases}$$

and this is a consequence of the assumption (3.5), where

$$A_{1,2}(t)=\frac{1}{2}(a_+(t)\pm a_+(-t)),\quad B_{1,2}(t)=\frac{1}{2}(b_+(t)\pm b_+(-t)),\quad C_{1,2}(t)=\frac{1}{2}(a_-(t)\pm a_-(-t)),$$

$$f_{1,2}^*(t)=\frac{1}{2}\left\{\left[f(t)-\sum_{j=1}^{m}\lambda_j a_j(t)\right]\pm\left[f(-t)-\sum_{j=1}^{m}\lambda_j a_j(-t)\right]\right\}.$$

Write $\varphi_1(t)=(Q_1\varphi)(t)$ and $\varphi_2(t)=(Q_2\varphi)(t),$ then $\varphi_j\in X_j$ for $j=1,2.$ Hence, we get the following system in $X_1\times X_2$:

$$\begin{cases}A_1(t)\varphi_1(t)+C_2(t)\varphi_2(t)+B_2(t)(S\varphi_1)(t)+(Q_1l\varphi_1)(t)=f_1^*(t),\\ A_2(t)\varphi_1(t)+C_1(t)\varphi_2(t)+B_1(t)(S\varphi_1)(t)+(Q_2l\varphi_1)(t)=f_2^*(t).\end{cases} \tag{3.6}$$

Lemma 3.1. *If* (φ_1,φ_2) *is a solution of the equation* (3.6) *in* $X\times X,$ *then* $(Q_1\varphi_1,Q_2\varphi_2)$ *is its solution in* $X_1\times X_2$.

Thus, it is enough to consider the system (3.6) in the space $X\times X$ only.

Now we consider the case $A_1(t)C_1(t)-A_2(t)C_2(t)\neq 0,\ \forall t\in\mathbb{R}$. Rewrite (3.6) in the form

$$\begin{cases}u(t)\varphi_1(t)+v(t)(S\varphi_1)(t)+Q_1[a_-(-t)(l\varphi_1)(t)]=C_1(t)f_1^*(t)-C_2(t)f_2^*(t),\\ u(t)\varphi_2(t)+v_1(t)(S\varphi_1)(t)+Q_2[a_+(-t)(l\varphi_1)(t)]=A_1(t)f_2^*(t)-A_2(t)f_1^*(t),\end{cases} \tag{3.7}$$

where

$$u(t)=\frac{1}{2}[a_+(-t)a_-(t)+a_+(t)a_-(-t)]=Q_1[a_+(t)a_-(-t)],$$

$$v(t)=\frac{1}{2}[b_+(t)a_-(-t)-a_-(t)b_+(-t)]=Q_2[b_+(t)a_-(-t)],$$

$$v_1(t)=\frac{1}{2}[a_+(t)b_+(-t)+a_+(-t)b_+(t)]=Q_1[a_+(t)b_+(-t)].$$

THEOREM 3.1. *Suppose that the function* $l(\tau,t)$ *satisfies the condition* (3.5), *i.e.* $l(-\tau,t)=l(\tau,t)$ *and*

$$[u(t)+v(t)]^{-1}Q_1[a_-(-t)(l\varphi_1)(t)]\in H_p^{++}, \tag{3.8}$$

then the equation (3.4) *admits all solutions in a closed form*

$$\varphi(t)=(Q_1\varphi_1)(t)+(Q_2\varphi_2)(t),$$

where $(\varphi_1(t),\varphi_2(t))$ *is a solution of the system* (3.7) *in* $X\times X$.

Proof. Put

$$\Phi_1(z)=\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{\varphi_1(\tau)}{\tau-z}d\tau,$$

according to Sokhotski-Plemelij formula, we have

$$\begin{cases}\varphi_1(t)=\Phi_1^+(t)-\Phi_1^-(t),\\ (S\varphi_1)(t)=\Phi_1^+(t)+\Phi_1^-(t).\end{cases} \tag{3.9}$$

Put

$$\begin{cases}\Phi^+(t)=\Phi_1^+(t)-\dfrac{Q_1[a_-(-t)l\Phi_1^-(t)]}{u(t)+v(t)}\in X^+,\\ \Phi^-(t)=\Phi_1^-(t),\end{cases} \tag{3.10}$$

we reduce equation the first equation of system (3.7) to the following boundary problem: find pairs of analytic functions on upper and lower half plane $\Phi^+(z),\Phi^-(z)$ and satisfies

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t), \tag{3.11}$$

where

$$G(t)=\frac{u(t)-v(t)}{u(t)+v(t)},\quad g(t)=\frac{C_1(t)f_1^*(t)-C_2(t)f_2^*(t)}{u(t)+v(t)}.$$

Suppose that $u^2(t)-v^2(t)$ is a non-vanishing on $\mathbb{R}$. Then $G(t),g(t)\in X$ and $G(t)\neq 0$ for any $t\in\mathbb{R}$. Put

$$\varkappa=IndG(t)=\frac{1}{2\pi i}\int_{\mathbb{R}}d\ln G(t),$$

$$\Gamma(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\ln\left[\left(\frac{\tau-i}{\tau+i}\right)^{-\varkappa}G(\tau)\right]\frac{d\tau}{\tau-z},$$

$$X^+(z)=e^{\Gamma^+(z)},X^-(z)=\left(\frac{z-i}{z+i}\right)^{-\varkappa}e^{\Gamma^-(z)}.$$

We have to consider the following cases:

1. If $\varkappa\geq 0$ then the problem (3.11) has general solution is given by formula:

$$\Phi(z)=X(z)\left[\Psi(z)+\frac{P_{\varkappa-1}(z)}{(z+i)^{\varkappa}}\right], \tag{3.12}$$

where

$$\Psi(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{g(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-z} \tag{3.13}$$

and

$$P_{\varkappa-1}(z)=p_1+p_2z+\cdots+p_{\varkappa}z^{\varkappa-1}, \tag{3.14}$$

is a polynomial of degree $\varkappa-1$ with arbitrary complex coefficients.

2. If $\varkappa < 0$ then the necessary condition for the problem (3.11) to be solvable is that

$$\int_{-\infty}^{+\infty} \frac{g(\tau)}{X^{+}(\tau)} \frac{d\tau}{(\tau+i)^{k}} = 0, k = 1, 2, \ldots, -\varkappa .$$

This condition can be written as follows

$$\int_{-\infty}^{+\infty} \frac{C_1(\tau) f_1^{*}(\tau) - C_2(\tau) f_2^{*}(\tau)}{X^{+}(\tau)(u(\tau)+v(\tau))} \frac{d\tau}{(\tau+i)^{k}} = 0, k = 1, 2, \ldots, -\varkappa . \tag{3.15}$$

If the condition (3.15) is satisfied then the solution is given by formula:

$$\Phi(z) = X(z)\Psi(z)$$

Hence, we have

$$\varphi_1(t) = \Phi_1^{+}(t) - \Phi_1^{-}(t) =$$

$$= \Phi^{+}(t) + \frac{Q_1[a\_(-t) l \Phi_1^{-}(t)]}{u(t)+v(t)} - \Phi^{-}(t)$$

and $\varphi_2(t)$ is defined by (3.7).

We have

$$\Psi(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{g(\tau)}{X^{+}(\tau)} \frac{d\tau}{\tau - z} = B(z) - \sum_{j=1}^{m} \lambda_j A_j(z),$$

where

$$B(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{1}{2} C_1(\tau)[f(\tau) + f(-\tau)] - \frac{1}{2} C_2(\tau)[f(\tau) - f(-\tau)]}{[u(\tau)+v(\tau)] X^{+}(\tau)} \cdot \frac{d\tau}{\tau - z},$$

$$A_j(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{1}{2} C_1(\tau)[a_j(\tau) + a_j(-\tau)] - \frac{1}{2} C_2(\tau)[a_j(\tau) - a_j(-\tau)]}{[u(\tau)+v(\tau)] X^{+}(\tau)} \cdot \frac{d\tau}{\tau - z}.$$

1. If $\varkappa \geq 0$ then $\Phi(z) = X(z)\left[\Psi(z) + \frac{P_{\varkappa-1}(z)}{(z+i)^{\varkappa}}\right]$ and

$$\varphi_1(t) = \Phi^{+}(t) + \frac{Q_1[a\_(-t) l \Phi^{-}(t)]}{u(t)+v(t)} - \Phi^{-}(t) = \omega(t) - \sum_{j=1}^{m} \lambda_j h_j(t) + \sum_{k=1}^{\varkappa} p_k e_k(t),$$

where

$$\omega(t) = X^{+}(t) B^{+}(t) - X^{-}(t) B^{-}(t) + \frac{Q_1\{a\_(-t) l [X^{-}(t) B^{-}(t)]\}}{u(t)+v(t)}, \tag{3.16}$$

$$h_j(t) = X^{+}(t) A_j^{+}(t) + \frac{Q_1\{a\_(-t) l [X^{-}(t) A_j^{-}(t)]\}}{u(t)+v(t)} - X^{-}(t) A_j^{-}(t), j = 1, \ldots, m, \tag{3.17}$$

$$e_k(t) = \frac{X^{+}(t) t^{k-1}}{(t+i)^{\varkappa}} + \frac{Q_1\left\{a\_(-t) l \left[\frac{X^{-}(t) t^{k-1}}{(t+i)^{\varkappa}}\right]\right\}}{u(t)+v(t)} - \frac{X^{-}(t) t^{k-1}}{(t+i)^{\varkappa}}, k = 1, \ldots, \varkappa , \tag{3.18}$$

$$(S\varphi_1)(t) = \Phi_1^+(t) + \Phi_1^-(t) = \Phi^+(t) + \frac{Q_1[a_-(-t)l\Phi^-(t)]}{u(t)+v(t)} + \Phi^-(t) =$$

$$= X^+(t)B^+(t) + X^-(t)B^-(t) + \frac{Q_1\{a_-(-t)l[X^-(t)B^-(t)]\}}{u(t)+v(t)} -$$

$$-\sum_{j=1}^{m}\lambda_j\{X^+(t)A_j^+(t) + \frac{Q_1\{a_-(-t)l[X^-(t)A_j^-(t)]\}}{u(t)+v(t)} + X^-(t)A_j^-(t)\} +$$

$$+\sum_{k=1}^{i} p_k\left\{\frac{X^+(t)t^{k-1}}{(t+i)^i} + \frac{Q_1\left\{a_-(-t)l\left[\frac{X^-(t)t^{k-1}}{(t+i)^i}\right]\right\}}{u(t)+v(t)} + \frac{X^-(t)t^{k-1}}{(t+i)^i}\right\}.$$

$$(l\varphi_1)(t) = l[\Phi_1^+(t) - \Phi_1^-(t)]) = -(l\Phi^-)(t) -$$

$$-l[X^-(t)B^-(t)] + \sum_{j=1}^{m}\lambda_j l[X^-(t)A_j^-(t)] - \sum_{k=1}^{i} p_k l\left[X^-(t)\frac{t^{k-1}}{(t+i)^i}\right].$$

From (3.7), we have

$$\varphi_2(t) = [u(t)]^{-1}\{A_1(t)f_2^*(t) - A_2(t)f_1^*(t) - v_1(t)(S\varphi_1)(t) - Q_2[a_+(-t)(l\varphi_1)(t)]\} =$$

$$= [u(t)]^{-1}\left\{\frac{1}{2}A_1(t)[f(t) - f(-t)] - \frac{1}{2}A_2(t)[f(t) + f(-t)] -\right.$$

$$\left.-v_1(t)\left[X^+(t)B^+(t) + X^-(t)B^-(t) + \frac{Q_1\{a_-(-t)l[X^-(t)B^-(t)]\}}{u(t)+v(t)}\right] + Q_2[a_+(-t)l(X^-(t)B^-(t))]\right\} -$$

$$-\sum_{j=1}^{m}\lambda_j[u(t)]^{-1}\left\{\frac{1}{2}A_1(t)[a_j(t) - a_j(-t)] - \frac{1}{2}A_2(t)[a_j(t) + a_j(-t)] -\right.$$

$$\left.-v_1(t)\left[X^+(t)A_j^+(t) + \frac{Q_1\{a_-(-t)l[X^-(t)A_j^-(t)]\}}{u(t)+v(t)} + X^-(t)A_j^-(t)\right] + Q_2[a_+(-t)l(X^-(t)A_j^-(t))]\right\} +$$

$$+\sum_{k=1}^{i} p_k[u(t)]^{-1}\left\{-v_1(t)\left\{\frac{X^+(t)t^{k-1}}{(t+i)^i} + \frac{Q_1\left\{a_-(-t)l\left[\frac{X^-(t)t^{k-1}}{(t+i)^i}\right]\right\}}{u(t)+v(t)} + \frac{X^-(t)t^{k-1}}{(t+i)^i}\right\} +\right.$$

$$\left.+Q_2\left\{a_+(-t)l\left[\frac{X^-(t)t^{k-1}}{(t+i)^i}\right]\right\}\right\} = \delta(t) - \sum_{j=1}^{m}\lambda_j\theta_j(t) + \sum_{k=1}^{i} p_k\xi_k(t),$$

where

$$\delta(t) = [u(t)]^{-1}\left\{\frac{1}{2}A_1(t)[f(t) - f(-t)] - \frac{1}{2}A_2(t)[f(t) + f(-t)] - v_1(t)\times\right.$$

$$\left.\times\left[X^+(t)A_j^+(t) + \frac{Q_1\{a_-(-t)l[X^-(t)A_j^-(t)]\}}{u(t)+v(t)} + X^-(t)A_j^-(t)\right] + Q_2[a_+(-t)l(X^-(t)A_j^-(t))]\right\},$$

$$\theta_j(t) = [u(t)]^{-1}\left\{\frac{1}{2}A_1(t)[a_j(t) - a_j(-t)] - \frac{1}{2}A_2(t)[a_j(t) + a_j(-t)] - v_1(t)\times\right.$$

$$\left.\times\left[X^+(t)A_j^+(t) + \frac{Q_1\{a_-(-t)l[X^-(t)A_j^-(t)]\}}{u(t)+v(t)} + X^-(t)A_j^-(t)\right] + Q_2[a_+(-t)l(X^-(t)A_j^-(t))]\right\},$$

$$j=1,\dots,m.$$

$$\xi_k(t)=[u(t)]^{-1}\left\{-v_1(t)\left\{\frac{X^+(t)t^{k-1}}{(t+i)^{\mathrm{i}}}+\frac{Q_1\left\{a_-(-t)l\left[\frac{X^-(t)t^{k-1}}{(t+i)^{\mathrm{i}}}\right]\right\}}{u(t)+v(t)}+\frac{X^-(t)t^{k-1}}{(t+i)\mathrm{i}}\right\}+\right.$$

$$\left.+Q_2\left\{a_+(-t)l\left[\frac{X^-(t)t^{k-1}}{(t+i)^{\mathrm{i}}}\right]\right\}\right\},\quad k=1,\dots,\mathrm{i}\ .$$

We have the solution of (3.2) is given by formula:

$$\varphi(t)=(Q_1\omega)(t)+(Q_2\delta)(t)-\sum_{j=1}^{m}\lambda_j[(Q_1h_j)(t)+(Q_2\theta_j)(t)]+\sum_{k=1}^{\mathrm{i}}p_k[(Q_1e_k)(t)+(Q_2\xi_k)(t)]. \tag{3.19}$$

Substituting (3.19) into the condition (3.3) , we obtain

$$\lambda_i=d_i-\sum_{j=1}^{m}\lambda_j e_{ij}+\sum_{k=1}^{\mathrm{i}}p_k g_{ik},\, i=1,\dots,m, \tag{3.20}$$

where

$$d_i=N_{b_i}((Q_1\omega)(t)+(Q_2\delta)(t)),$$

$$e_{ij}=N_{b_i}((Q_1h_j)(t)+(Q_2\theta_j)(t)),$$

$$g_{ik}=N_{b_i}(Q_1e_k)(t)+(Q_2\xi_k)(t).$$

Put

$$\Lambda=\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_m\end{pmatrix}_{m\times 1},\quad P=\begin{pmatrix}p_1\\ p_2\\ \vdots\\ p_{\mathrm{i}}\end{pmatrix}_{\mathrm{i}\times 1},\quad D=\begin{pmatrix}d_1\\ d_2\\ \vdots\\ d_m\end{pmatrix}_{m\times 1},\quad E=\begin{pmatrix}e_{11}&e_{12}&\cdots&e_{1m}\\ e_{21}&e_{22}&\cdots&e_{2m}\\ \vdots&\vdots&\ddots&\vdots\\ e_{m1}&e_{m2}&\cdots&e_{mm}\end{pmatrix}_{m\times m},$$

$$G=\begin{pmatrix}g_{11}&g_{12}&\cdots&g_{1\mathrm{i}}\\ g_{21}&g_{22}&\cdots&g_{2\mathrm{i}}\\ \vdots&\vdots&\ddots&\vdots\\ g_{m1}&g_{m2}&\cdots&g_{m\mathrm{i}}\end{pmatrix}_{m\times\mathrm{i}}.$$

Now we write (3.20) in the form of matrix condition

$$(I+E)\Lambda=D-GP, \tag{3.21}$$

where I is the unit matrix.

From (3.21) we can say that the function φ determined by (3.19) is a solution of (3.1) if and only if $(\lambda_1,\lambda_2,\dots,\lambda_m)$ satisfies the following matrix condition

$$(I+E)\Lambda=D-GP.$$

2. If $\mathrm{i}<0$, then the equation (3.2) has solutions if and only if the condition (3.15) satisfied. If this is in case, then $p_{\mathrm{i}-1}(t)\equiv 0$. So, all solutions of (3.2) are given by

$$\varphi(t)=(Q_1\varphi_1)(t)+(Q_2\varphi_2)(t), \tag{3.22}$$

where

$$\varphi_1(t)=\omega(t)-\sum_{j=1}^{m}\lambda_j h_j(t),$$

and $\omega(t), h_j(t)$ are determined by (3.16)

$$\varphi_2(t) = \delta(t) - \sum_{j=1}^{m} \lambda_j \theta_j(t).$$

Hence, the solution φ determined by (3.22) is a solution of the equation (3.1) if $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ satisfies the following matrix condition

$$(I + E)\Lambda = D. \tag{3.23}$$

On the other hand, the condition (3.15) is of the form:

$$d'_k - \sum_{j=1}^{m} \lambda_j e'_{kj} = 0, k = 1, 2, \ldots, -\text{i} , \tag{3.24}$$

where

$$d'_k = \int_{-\infty}^{+\infty} \frac{\frac{1}{2} C_1(\tau)[f(\tau) + f(-\tau)] - \frac{1}{2} C_2(\tau)[f(\tau) - f(-\tau)]}{[u(\tau) + v(\tau)] X^{+}(\tau)} \cdot \frac{d\tau}{(\tau + i)^k},$$

$$e'_{kj} = \int_{-\infty}^{+\infty} \frac{\frac{1}{2} C_1(\tau)[a_j(\tau) + a_j(-\tau)] - \frac{1}{2} C_2(\tau)[a_j(\tau) - a_j(-\tau)]}{[u(\tau) + v(\tau)] X^{+}(\tau)} \cdot \frac{d\tau}{(\tau + i)^k}.$$

Put

$$D' = \begin{pmatrix} d'_1 \\ d'_2 \\ \vdots \\ d'_{-\text{i}} \end{pmatrix}_{-\text{i} \times 1}, \qquad E' = \begin{pmatrix} e'_{11} & e'_{12} & \cdots & e'_{1m} \\ e'_{21} & e'_{22} & \cdots & e'_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ e'_{-\text{i}\,1} & e'_{-\text{i}\,2} & \cdots & e'_{-\text{i}\,m} \end{pmatrix}_{-\text{i} \times m}.$$

We write (3.24) in the form of matrix condition

$$E'\Lambda = D'. \tag{3.25}$$

Combining (3.23) and (3.25) we can say that the function φ determined by (3.22) is a solution of (3.1) if and only if $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ satisfy the following matrix condition

$$\begin{pmatrix} I + E \\ E' \end{pmatrix}_{(m-\text{i}) \times m} \times \Lambda = \begin{pmatrix} D \\ D' \end{pmatrix}_{(m-\text{i}) \times 1}.$$

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