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SHORT COMMUNICATIONS

On the Exceptional Case of the Characteristic Singular Equation with Cauchy Kernel

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Abstract—We study the exceptional case of the characteristic singular integral equation with Cauchy kernel in which its coefficients admit zeros or singularities of complex orders at finitely many points of the contour. By reduction to a linear conjugation problem, we obtain an explicit solution formula and solvability conditions for this equation in weighted Hölder classes.

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Consider the characteristic singular integral equation with Cauchy kernel [1, p. 176]

$$a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i}\int\limits_\Gamma \frac{\varphi(t)\,dt}{t-t_0} = f(t_0), \qquad t_0 \in \Gamma, \tag{1}$$

on a simple piecewise smooth closed contour Γ with counterclockwise orientation. Recall that Eq. (1) is said to be of normal type if the coefficients $(a+b)(t)$ and $(a-b)(t)$ belong to the class $H(\Gamma)$ and are invertible in that class. The exceptional case of this equation occurs if the functions $a \pm b$ may have zeros at finitely many points of the contour Γ. Let $F \subseteq \Gamma$ be a finite set containing all corner points of the contour and split into two subsets F_j, $j = 1, 2$, and let the coefficients a and b of the equation satisfy the conditions

$$(a+b)(t) = O(|t-\tau|^{\alpha_\tau}), \qquad t \to \tau \in F_1, \qquad (a-b)(t) = O(|t-\tau|^{\beta_\tau}), \qquad t \to \tau \in F_2, \tag{2}$$

where α_τ and β_τ are given sets of complex numbers.

Equation (1) with integer α_τ and β_τ was considered under these assumptions by numerous authors. In particular, it was completely studied in [2, 3] by reduction to the Riemann boundary value problem (the linear conjugation boundary value problem). Independently of these papers, the same problem was studied in [4, 5] by a different method under the condition that only one of the functions $a(t) \pm b(t)$ has zeros of integer orders on the contour Γ.

As was mentioned above, Eq. (1) can be reduced to the corresponding linear conjugation problem in a standard way [1, pp. 176–177]. The special case

$$\Phi^+(t) = |t-\tau_1|^\alpha G(t)\Phi^-(t) + g(t) \tag{3}$$

of the linear conjugation problem, where τ_1 is a given point of the contour Γ, $\alpha \in \mathbb{C}$, the coefficients $G(t)$ and $g(t)$ satisfy the Hölder condition, and $G(t) \neq 0$, was considered in [6]. Note that Eq. (1) can be reduced to problem (3) under the assumption that one of the functions $(a \pm b)(t)$ has one zero of arbitrary order on the contour Γ.

The boundary value linear conjugation problem

$$\Phi^+(t) - \prod_{\tau \in F}(t-\tau)^{\alpha_\tau} G(t)\Phi^-(t) = g(t), \qquad t \in \Gamma, \tag{4}$$

on a simple smooth closed contour Γ with complex exponents α_τ was studied in [7] in weighted Hölder classes with an arbitrary power-law weight. A closed-form representation of the solution was obtained there, and the problem solvability conditions were described. The present paper continues the paper [7] and uses reduction to the boundary value problem (4) to study the exceptional case of Eq. (1) in which all exponents α_τ and β_τ occurring in conditions (2) are complex, and Γ is a simple piecewise Lyapunov closed contour.

By $H_0(\Gamma, F)$ we denote the class of piecewise Hölder functions on Γ with possible simple discontinuities at the points $\tau \in F$. More precisely, this class consists of functions $h(t)$ satisfying the Hölder condition on any arc $\Gamma_0 \subseteq \Gamma$ that does not contain $\tau \in F$ as interior points. The one-sided limit values of h at the points $\tau \in F$ with regard to the chosen orientation of the contour will be denoted by $h(\tau \pm 0)$.

The contour Γ divides the complex plane into the interior domain D^+ and the exterior domain D^-. In these domains, consider the analytic functions

$$A(z) = \prod_{\tau \in F_1} (z-\tau)^{\alpha_\tau}, \qquad z \in D^+, \qquad B(z) = \prod_{\tau \in F_2} \left(\frac{z-\tau}{z-z_0}\right)^{\beta_\tau}, \qquad z \in D^-, \tag{5}$$

where $z_0 \in D^+$ is a fixed point and the branches of power-law functions in the second relation are chosen so as to ensure that they tend to unity as $z \to \infty$. For the boundary values of these functions on the contour Γ, we use the same notation.

In this notation, we refine conditions (2) by requiring that the factors $c_\pm$ occurring in the representations

$$a + b = c_+ A, \qquad a - b = c_- B \tag{6}$$

belong to the class $H_0(\Gamma, F)$ and are invertible in that class.

By $\mathring{H}(\Gamma, F)$ we denote the class of functions $\varphi \in H(\Gamma)$ vanishing on F. Let us introduce the weight function

$$\varrho_\lambda(t) = \prod_{\tau \in F} |t-\tau|^{\lambda_\tau}, \tag{7}$$

where $\lambda = (\lambda_\tau,\ \tau \in F)$. By definition, the class $\mathring{H}_\lambda(\Gamma, F)$ consists of functions φ such that $\varrho_{-\lambda}\varphi \in \mathring{H}(\Gamma, F)$. In a similar way, one can define the classes $\mathring{H}(\overline{D}^\pm, F)$ and $\mathring{H}_\lambda(\overline{D}^\pm, F)$ of analytic functions in $D^\pm$ with the only difference that the weight functions are given by

$$R_\lambda(z) = \prod_{\tau \in F} (z-\tau)^{\lambda_\tau}, \qquad z \in D^+, \qquad R_\lambda(z) = \prod_{\tau \in F} \left(\frac{z-\tau}{z-z_0}\right)^{\lambda_\tau}, \qquad z \in D^-.$$

In what follows, we assume that Γ is a piecewise Lyapunov contour; thus, any branch of $\arg(t-\tau)$ continuous on $\Gamma \backslash \tau$ belongs to the class $H_0(\Gamma, F)$ with a discontinuity at the point $\tau \in F$. One can readily see that the class $\mathring{H}(\Gamma, F)$ is invariant under the multiplication by the bounded functions $|t-\tau|^{i\alpha}$ and $e^{\beta \arg(t-\tau)}$ with arbitrary $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. In particular, in notation (5) and (7), the class $\mathring{H}(\Gamma, F)$ is invariant under the multiplication by bounded functions $A(t)\varrho_{\nu^+}^{-1}(t)$ and $B(t)\varrho_{\nu^-}^{-1}(t)$, $t \in \Gamma$, which are defined by the weight orders

$$\nu_\tau^+ = \begin{cases} \operatorname{Re}\alpha_\tau & \text{for} \quad \tau \in F_1, \\ 0 & \text{for} \quad \tau \in F_2, \end{cases} \qquad \nu_\tau^- = \begin{cases} 0 & \text{for} \quad \tau \in F_1, \\ \operatorname{Re}\beta_\tau & \text{for} \quad \tau \in F_2. \end{cases} \tag{8}$$

We introduce one more weight order

$$\nu = \max(\nu^+, \nu^-), \qquad \nu_\tau = \begin{cases} \max(\operatorname{Re}\alpha_\tau, 0) & \text{for} \quad \tau \in F_1, \\ \max(\operatorname{Re}\beta_\tau, 0) & \text{for} \quad \tau \in F_2. \end{cases} \tag{9}$$

We seek a solution of Eq. (1) in the class $\mathring{H}_\lambda(\Gamma, F)$, $-1 \le \lambda < 0$, under the assumption that its right-hand side belongs to the class $\mathring{H}_{\lambda+\nu}(\Gamma, F)$.

It is well known (e.g., see [8, p. 50]) that if $\varphi(t)$ belongs to the class $\mathring{H}_\lambda(\Gamma, F)$, $-1 \le \lambda < 0$, then the Cauchy type integral

$$\phi = I\varphi, \qquad (I\varphi)(z) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{\varphi(t)\,dt}{t-z}, \tag{10}$$

belongs to the class $H_\lambda(\overline{D}^{\pm}, F)$ and vanishes at infinity, and the Sokhotski–Plemelj formulas

$$2\phi^{\pm} = \pm\varphi + S\varphi \tag{11}$$

hold; here and in the following, S is the Cauchy singular operator

$$(S\varphi)(t_0) = \frac{1}{\pi i}\int\limits_\Gamma \frac{\varphi(t)\,dt}{t-t_0}, \qquad t_0 \in \Gamma.$$

As a result, Eq. (1) can be reduced (e.g., see [1, p. 176]) to the linear conjugation problem

$$(a+b)\phi^+ - (a-b)\phi^- = f; \tag{12}$$

moreover, the relation between them is given by the integral (10). The substitution

$$\psi(z) = \begin{cases} A(z)\phi(z) & \text{for} \quad z \in D^+, \\ B(z)\phi(z) & \text{for} \quad z \in D^-, \end{cases} \tag{13}$$

takes the function $\phi \in \mathring{H}_\lambda(\overline{D}^{\pm}, F)$ to the function $\psi \in \mathring{H}_{\lambda+\nu^\pm}(\overline{D}^{\pm}, F)$, which, by (11), satisfies the boundary condition

$$\psi^+ - G\psi^- = g, \tag{14}$$

where we have set $G = c_-/c_+$ and $g = f/c_+$ in notation (6). Since $\mathring{H}_{\lambda+\nu} \subseteq \mathring{H}_{\lambda+\nu^\pm}$, we find that it suffices to solve problem (14) in the class $\mathring{H}_{\lambda+\nu}$.

To this end, we consider the canonical function $X(z)$ of the problem related to that class [9]. Take an arbitrary continuous branch of the logarithm $(\ln G)(t)$ on $\Gamma\backslash F$, which obviously belongs to $H_0(\Gamma, F)$, and set

$$\delta_\tau = \frac{(\ln G)(\tau - 0) - (\ln G)(\tau + 0)}{2\pi i}. \tag{15}$$

Note that the sum of these numbers over $\tau \in F$ coincides with the sum of increments of the corresponding function on the arcs forming the contour,

$$\sum_\tau \delta_\tau = \frac{1}{2\pi i}(\ln G)|_\Gamma. \tag{16}$$

In notation (10), consider the function $Y = \exp[I(\ln G)]$, which, according to well-known properties of the Cauchy type integral [8, p. 50], can be represented in the form

$$Y(z) = Y_0(z)\prod_{\tau\in F}(z-\tau)^{-\delta_\tau}, \qquad z \in D^+, \qquad Y(z) = Y_0(z)\prod_{\tau\in F}\left(\frac{z-\tau}{z-z_0}\right)^{-\delta_\tau}, \qquad z \in D^-,$$

where the functions Y_0 belong to the class $H_0(\overline{D}^{\pm}, F)$ and are invertible in it. The canonical function $X(z)$ is defined by the formula

$$X(z) = Y(z)\prod_{\tau\in F}(z-\tau)^{-n_\tau}, \tag{17}$$

where the integers n_τ satisfy the condition $\lambda_\tau+\nu_\tau<-\operatorname{Re}\delta_\tau-n_\tau<0$. This inequality is equivalent to $n_\tau=-[\operatorname{Re}\delta_\tau+1+\lambda_\tau+\nu_\tau]$, where $[x]$ is the integer part of a real number x. By (17), the function $X(z)$ has the following property at infinity:

$$\lim_{z\to\infty} z^{\varkappa}X(z)=1, \qquad \varkappa=-\sum_{\tau}[\operatorname{Re}\delta_\tau+1+\lambda_\tau+\nu_\tau]. \tag{18}$$

Note that $\varkappa$ is independent of the choice of the branch $\ln G$. Indeed, if $\widetilde{\ln}\, G$ is another branch and the $\widetilde{\delta}_\tau$ are defined for it by analogy with (15), then $\widetilde{\delta}_\tau=\delta_\tau+m_\tau$ with some integers m_τ; moreover, by (16), the sum $\sum_\tau m_\tau$ is zero, so that the replacement of δ_τ with $\widetilde{\delta}_\tau$ in (18) does not change the value of $\varkappa$.

Let P_n, $n\ge 0$, be the class of polynomials of degree $\le n$; we set $P_n=0$ for $n<0$. For brevity, we introduce the bilinear form

$$\langle\varphi,\psi\rangle=\int\limits_{\Gamma}\varphi(t)\psi(t)\,dt.$$

In this notation, the solvability of problem (14) in the class of functions $\psi(z)\in\mathring{H}_{\lambda+\nu}$ vanishing at infinity is described by the following assertion [9].

Theorem 1. *The general solution of problem* (14) *in the class* $\mathring{H}_{\lambda+\nu}$ *has the form*

$$\psi(z)=X(z)\left[I\left(\frac{g}{X^+}\right)+p\right], \qquad p\in P_{\varkappa-1}, \tag{19}$$

where g satisfies the orthogonality conditions

$$\left\langle g,\frac{q}{X^+}\right\rangle=0, \qquad q\in P_{-\varkappa-1}. \tag{20}$$

If $\varkappa\ge 0$, then the orthogonality conditions (20) are unnecessary; if $\varkappa\le 0$, then the polynomials p in the representation (19) are zero; i.e., the solution is unique. In both cases, the index of the problem is $\varkappa$.

By returning from ψ to the function ϕ in (13), as a result, one can write out the solutions $\varphi=\phi^+-\phi^-$ of the original equation (1). By the Sokhotski–Plemelj formulas applied to the function (19), we have

$$\phi^+=\frac{X^+}{A}\left[\frac{g}{2X^+}+\frac{1}{2}S\left(\frac{g}{X^+}\right)+p\right], \qquad \phi^-=\frac{X^-}{A}\left[-\frac{g}{2X^+}+\frac{1}{2}S\left(\frac{g}{X^+}\right)+p\right],$$

whence it follows that

$$\varphi=\left(\frac{1}{2A}+\frac{1}{2BG}\right)g+\left(\frac{1}{2A}-\frac{1}{2BG}\right)X^+\left[S\left(\frac{g}{X^+}\right)\right]+X^+\left(\frac{1}{A}-\frac{1}{BG}\right)p. \tag{21}$$

By (6), we have

$$\frac{a+b}{a-b}=\frac{BG}{A};$$

thus,

$$\frac{1}{A}+\frac{1}{BG}=\frac{2a}{A(a-b)}, \qquad \frac{1}{A}-\frac{1}{BG}=-\frac{2b}{A(a-b)}. \tag{22}$$

In a similar way, we obtain the relation

$$g=\frac{Af}{a+b}. \tag{23}$$

By setting

$$\varrho = \frac{(a+b)X^+}{A}$$

for brevity, by denoting $-2p$ again by p, and by taking into account relations (22) and (23), we reduce formula (21) to the form

$$(a^2 - b^2)\varphi = af - bS_* f + \varrho p, \qquad p \in P_{\varkappa-1}, \tag{24}$$

with the weighted singular operator $S_* f = \varrho[S(\varrho^{-1} f)]$. Accordingly, we rewrite the orthogonality condition (20) in the form

$$\left\langle f, \frac{q}{\varrho} \right\rangle = 0, \qquad q \in P_{-\varkappa-1}. \tag{25}$$

We have thereby proved the following assertion.

Theorem 2. *Let the function f belong to $\overset{\circ}{H}_{\lambda+\nu}(\Gamma, F)$, and let the orthogonality condition* (25) *be satisfied. Then formula* (24) *describes the solutions of Eq.* (1) *in the class $\overset{\circ}{H}_\lambda(\Gamma, F)$.*

Note that if Eq. (1) is considered in the widest class $\overset{\circ}{H}_{-1}(\Gamma, F)$, then the expression (18) for the index $\varkappa$ acquires the form

$$\varkappa = -\sum_\tau [\,\mathrm{Re}\,\delta_\tau + \nu_\tau].$$

In particular, if the real parts of all exponents α_τ and β_τ are nonnegative, then, by relation (9), the weight order ν is zero, and Theorem 2 describes the solvability of Eq. (1) with right-hand side $f \in \overset{\circ}{H}_{-1}$ in the same class with the index

$$\varkappa = -\sum_\tau [\,\mathrm{Re}\,\delta_\tau].$$

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