МАТЕМАТИКА

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CONVOLUTIONS OF THE FOURIER-COSINE AND FOURIER-SINE INTEGRAL TRANSFORMS AND INTEGRAL EQUATIONS OF THE CONVOLUTION TYPE*

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This paper introduces general definitions of convolutions without and with weight, obtains four new convolutions and generalized convolutions of the Fourier-cosine and Fourier-sine integral transforms. Furthermore, the paper investigates into a class of integral equations with the mixed Toeplitz-Hankel kernel. Namely, by using the constructed convolutions the explicit solutions are obtained.

1. Introduction and summary of results

The integral transforms of Fourier type and their convolutions have been studied for a long time ago, and they are applied in many fields of mathematics. Generalized convolutions for integral transforms and theirs applications were first studied by Churchill in 1940, then an idea of construction of the convolutions was formulated by Vilenkin in 1958 (see [4, 20]). There is an extensive list of materials concerning the applications of the integral transforms and of their convolutions (see [2, 5, 13, 17, 21]).

One knows that there are several relations, explicit or implicit, between the integral transforms of Cauchy, Fourier, Hankel, Laplace, Mellin (see [13, 17]). In recent years, many papers devoted to those transforms are given the convolutions, generalized convolutions, polyconvolutions and theirs applications (see [2, 3, 14, 15, 16, 18]). A reason that the theory of integral transforms and their convolutions attracts a lot of attention is that each of convolutions, generally speaking, is a new transform which can become an object of study (see [3, 12, 15, 18, 19]).

It is well-known that the Fourier-cosine and Fourier-sine integral transforms defined as follows:

$$(T_c f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \cos xy f(y)dy := g_c(x), \tag{1.1}$$

and

$$(T_c f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \sin xy f(y)dy := g_s(x), \tag{1.2}$$

where $\cos xy := \cos(x_1y_1 + \ldots + x_dy_d)$, $\sin xy := \sin(x_1y_1 + \ldots + x_dy_d)$. Remark that for any $f \in L^1(\mathbb{R}^d)$, the functions $(T_c f)(x)$, $(T_s f)(x)$ exist for every $x \in \mathbb{R}^d$ and they are the continuous functions vanishing at infinity (see [1, 11, 13, 17]).

The main purpose of this paper is to present some general definitions of convolutions, construct convolutions and generalized convolutions with and without weight-function for transforms T_c, T_s, F (F denoted the Fourier transform) and to solve a class of integral equations of the convolution type in $L^1(\mathbb{R}^d)$.

The paper is divided into three sections and organized as follows.

Section 2 is divided into two subsections. In Subsection 2.1, there are the general definitions of convolutions with and without weight for linear operators mapping from a linear space U to a commutative algebra V. In Subsection 2.2, there are four generalized convolutions for the transforms T_c, T_s. Generally speaking, each of convolutions is a new transform which can become an object of study. As usual, there exist difierent generalized convolutions for the same transform.

In Subsection 3.1, by using each of the obtained convolutions we construct the normed ring structures for $L_1(\mathbb{R}^d)$. In Subsection 3.2, we solve the integral equations with the mixed Toeplitz-Hankel kernel by the use of the constructed convolutions in Section 2. In particular, we obtain the explicit solutions in $L_1(\mathbb{R}^d)$ of the integral equations with the mixed Toeplitz-Hankel kernel.

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2. Generalized convolutions for T_c, T_s

This section contains two subsections. The general definitions of convolutions are in Subsection 2.1, and the generalized convolutions are in Subsection 2.2.

2.1. General definitions of convolutions. In 1967, the construction methods for convolutions and generalized convolutions for arbitrary integral transforms were proposed by Kakichev, and in 1990 a concept of the generalized convolutions for arbitrary linear operators was first introduced (see [7]). However, for the integral transforms some results of convolutions and generalized convolutions were obtained in 1997 (see [8]), and the generalization of these results were presented in 1998 (see [9]).

This subsection also introduces some definitions of convolutions and generalized convolutions for arbitrary linear operators from a linear space to a commutative algebra on the same field of scalars.

Let U be a linear space and let V be a commutative algebra on the field $\mathcal{K}$.

Let $T \in L(U, V)$ be a linear operator from U to V:

Definition 2.1. A bilinear map $* : U \times U : \to U$ is called a convolution for T, if $T(*(f, g)) = T(f)T(g)$ for any $f, g \in U$: The image $*(f, g)$ is denoted by $f \underset{T}{*} g$.

Let γ be an element in algebra V.

Definition 2.2. A bilinear map $* : U \times U : \to U$ is called the convolution with weight-element γ for T, if $T(*(f, g)) = \gamma T(f)T(g)$ for any $f, g \in U$: The image $*(f, g)$ is denoted by $f \overset{\gamma}{\underset{T}{*}} g$.

Each of the identities in Definitions 2.1, 2.2 is called the factorization identity (see [2, 9, 12, 16, 18]).

Let U_1, U_2, U_3 be the linear spaces on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are the linear operators from U_1, U_2, U_3 to V respectively.

Definition 2.3. A bilinear map $* : U_1 \times U_2 : \to U_3$ is called the convolution with weight-element γ for K_3, K_1, K_2 (that in order) if $K_3(*(f, g)) = \gamma K_1(f)K_2(g)$ for any $f \in U_1$, $g \in U_2$.

The image $*(f, g)$ is denoted by $f \overset{\gamma}{\underset{K_3, K_1, K_2}{*}} g$. If γ is unit of V, we say briefly the convolution for K_3, K_1, K_2. If $U_1 = U_2 = U_3$ and $K_1 = K_2 = K_3$, the convolution is denoted simply $f \overset{\gamma}{\underset{K_1}{*}} g$, and $f \underset{K_1}{*} g$ if is unit of V (see [6, 14]).

Remark 2.1. If operator K_3 is injective, the convolution $f \overset{\gamma}{\underset{K_3, K_1, K_2}{*}} g$ is formal determined, since $f \overset{\gamma}{\underset{K_3, K_1, K_2}{*}} g = K_3^{-1}(\gamma K_1(f)K_2(g))$ for any $f \in U_1$, $g \in U_2$.

In the next subsections, we consider $U_k = L^1(\mathbb{R}^d)$ ($k = 1, 2, 3$) with the integral by Lebesgue's mean, and V the algebra of all functions (real or complex) defined on $\mathbb{R}^d$.

2.2. Convolutions for T_c, T_s

There are four convolutions for T_c, T_s in this subsection.

Theorem 2.1. *If* $f, g \in L^1(\mathbb{R}^d)$, *then*

$$(f \underset{T_c}{*} g)(x) = \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} [f(x-u) + f(x+u)] g(u)du \tag{2.1}$$

defines the convolution for T_c, and the factorization identity is

$$T_c(f \underset{T_c}{*} g)(x) = (T_c f)(x)(T_c g)(x).$$

Proof. We prove $T_c(f \underset{T_c}{*} g) \in L^1(\mathbb{R}^d)$. We have

$$\int_{\mathbb{R}^d} \left|(f \underset{T_c}{*} g)(x)dx\right| = \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [f(x-u) + f(x+u)] g(u)du \right| dx \le$$

$$\le \frac{1}{2(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)|du \left[\int_{\mathbb{R}^d} |f(x-u)|dx + \int_{\mathbb{R}^d} |f(x+u)|dx \right] \le \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(u)|du \int_{\mathbb{R}^d} |f(x)|dx < +\infty .$$

We now prove the factorization identity. We have

$$(T_c f)(x)(T_c g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\cos xu\cos xv f(u)g(v)dudv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\cos x(u+v)+\cos x(u-v)]f(u)g(v)dudv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\cos xt[f(t-y)+f(t+y)]g(y)g(v)dydt = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\cos xt(f\underset{T_c}{*}g)(t)dt = T_c(f\underset{T_c}{*}g)(x).$$

The theorem is proved.

Corollary 2.1. *We have*

(i) $$(f\underset{T_c}{*}g)(x) = \frac{1}{2}\left[(f\underset{F}{*}g)(x) + (f\underset{F}{*}\check{g})(x)\right],$$

(ii) $$(f\underset{F}{*}g)(x) = \frac{1}{2}\left[(f\underset{T_c}{*}g)(x) + (f\underset{T_c}{*}\check{g})(x)\right].$$

Theorem 2.2. *If* $f, g \in L^1(\mathbb{R}^d)$ *, then*

$$(f\underset{T_c,T_s,T_s}{*}g)(x) = \frac{1}{2(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[-f(x-u)+f(x+u)]g(u)du \tag{2.2}$$

defines the convolution of for T_c*, and the factorization identity is*

$$T_c(f\underset{T_c,T_s,T_s}{*}g)(x) = (T_s f)(x)(T_s g)(x).$$

Proof. The fact that $f\underset{T_c,T_s,T_s}{*}g \in L^1(\mathbb{R}^d)$ is proved similarly as the proof of

Theorem 2.1. We prove the factorization identity. We have

$$(T_s f)(x)(T_s g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sin xu\sin xv f(u)g(v)dudv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[-\cos x(u+v)+\cos x(u-v)]f(u)g(v)dudv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\cos xt[-f(t-y)+f(t+y)]g(y)dydt =$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\cos xt(f\underset{T_c,T_s,T_s}{*}g)(t)dt = T_c(f\underset{T_c,T_s,T_s}{*}g)(x).$$

The theorem is proved.

Corollary 2.2. We have

(i) $$(f\underset{T_c,T_s,T_s}{*}g)(x) = \frac{1}{2}\left[(f\underset{F}{*}g)(x) + (f\underset{F}{*}\check{g})(x)\right],$$

(ii) $$(f\underset{F}{*}g)(x) = \frac{1}{2}\left[(f\underset{T_c,T_s,T_s}{*}g)(x) + (f\underset{T_c}{*}\check{g})(x)\right].$$

Theorem 2.3. *If* $f, g \in L^1(\mathbb{R}^d)$*, then*

$$(f\underset{T_s,T_c,T_s}{*}g)(x) = \frac{1}{2(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[f(x-u)-f(x+u)]g(u)du \tag{2.3}$$

defines the convolution for T_s T_c, T_s, and the factorization identity is

$$T_s(f \underset{T_s,T_c,T_s}{*} g)(x) = (T_c f)(x)(T_c g)(x).$$

Proof. It sufices to prove the factorization identity. We have

$$(T_c f)(x)(T_c g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\cos xu \sin xv\, f(u)g(v)\,du\,dv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\sin x(u+v) - \sin x(u-v)]\, f(u)g(v)\,du\,dv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sin xt\,[f(t-y) - f(t+y)]g(y)\,dy\,dt =$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\sin xt\,(f \underset{T_s,T_c,T_s}{*} g)(t)\,dt = T_s(f \underset{T_s,T_c,T_s}{*} g)(x).$$

The theorem is proved.

Corollary 2.3. *We have*

(i) $$(f \underset{T_s,T_c,T_s}{*} g)(x) = \frac{1}{2}\left[(f \underset{F}{*} g)(x) - (f \underset{F}{*} \breve{g})(x)\right];$$

(ii) $$(f \underset{F}{*} g)(x) = \frac{1}{2}\left[(f \underset{T_s,T_c,T_s}{*} g)(x) - (f \underset{T_s,T_c,T_s}{*} \breve{g})(x)\right].$$

Theorem 2.4. *If* $f, g \in L^1(\mathbb{R}^d)$, *then*

$$(f \underset{T_s,T_s,T_c}{*} g)(x) = \frac{1}{2(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[f(x-u) + f(x+u)]g(u)\,du \qquad (2.4)$$

defines the convolution for T_s, T_s, T_c, and the factorization identity is

$$T_s(f \underset{T_s,T_s,T_c}{*} g)(x) = (T_s f)(x)(T_c g)(x).$$

Proof. It suffices to prove the factorization identity. We have

$$(T_s f)(x)(T_c g)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sin xu \cos xv\, f(u)g(v)\,du\,dv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\sin x(u+v) + \sin x(u-v)]\, f(u)g(v)\,du\,dv =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sin xt\,[f(t-y) + f(t+y)]g(y)\,dy\,dt =$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\sin xt\,(f \underset{T_s,T_s,T_c}{*} g)(t)\,dt = T_s(f \underset{T_s,T_s,T_c}{*} g)(x).$$

The theorem is proved.

Corollary 2.4. *We have*

(i) $$(f \underset{T_s,T_s,T_c}{*} g)(x) = \frac{1}{2}\left[(f \underset{F}{*} g)(x) - (f \underset{F}{*} \breve{g})(x)\right];$$

(ii) $$(f \underset{F}{*} g)(x) = \frac{1}{2}\left[(f \underset{T_s,T_s,T_c}{*} g)(x) + (f \underset{T_s,T_s,T_c}{*} \breve{g})(x)\right].$$

Remark 2.2. The non-triviality of the convolutions in this subsection is proved as follows. Transforms T_c and T_s are the linear maps defined on $X := L^1(\mathbb{R}^d)$ .We see that $X \setminus \ker T_c \neq 0$, and $X / \ker T_s \neq 0$. For convolutions (2.1); (2.2), and (2.3); (2.4) we choose $f, g \in X \setminus \ker T_c$; $f, g \in X \setminus \ker T_s$, and $f \in X \setminus \ker T_c$, $g \in X \setminus \ker T_s$; $f \in X \setminus \ker T_s$, $g \in X / \ker T_c$, respectively. The non-triviality of each of the convolutions now follows from its factorization identity.

3. Applications

3.1. Normed ring structures on $L^1(\mathbb{R}^d)$

Definition 3.1. (see Naimark [10]) A vector space V with a ring structure and a vector norm is called the normed ring if $\|\upsilon\omega\| \le \|\upsilon\|\|\omega\|$, for all $\upsilon, \omega \in V$.

If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Let X denote the linear space $L^1(\mathbb{R}^d)$. Now we define norms for $f \in X$. For convolutions (2.1), (2.2), (2.3), (2.4), the norm is

$$\|f\| = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|f(x)|dx.$$

Theorem 3.1. *X, equipped with each of the convolution multiplications listed above, becomes a non-commutative normed ring having no unit.*

Proof. The proof for the first statement is divided into two steps.

Step 1. X has a normed ring structure. It is clear that *X*, equipped with each of the above listed convolution multiplications, has a ring structure. We have to prove the multiplicative inequality. We now prove that for convolution (2.3), the proof that for the others is similar. We have

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\left|(f \underset{T_s,T_c,T_s}{*} g)(x)\right|dx \le \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-u)-f(x+u)||g(u)|dudx \le$$

$$\le \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x-u)||g(u)|dxdu + \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|f(x+u)||g(u)|dxdu =$$

$$= \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}|g(u)|du\int_{\mathbb{R}^d}|f(x-u)|dx + \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}|g(u)|du\int_{\mathbb{R}^d}|f(x+u)|dx =$$

$$= \frac{1}{2}\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|g(u)|du\right)\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|f(x-u)|d(x-u)\right)+$$

$$+\frac{1}{2}\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|g(u)|du\right)\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}|f(x+u)|d(x+u)\right) = \|f\|\cdot\|g\|.$$

Thus $\left\|f \overset{\gamma_3}{\underset{T_1}{*}} g\right\| \le \|f\|\cdot\|g.\|$

Step 2. X has no unit. For briefness of our proof, let us use the common symbols: * for the convolutions listed above. First, we prove $T_s\Phi_0 \equiv 0$ and $T_c\Phi_0 = \Phi_0$, where $\Phi_0(x) = e^{-\frac{1}{2}|x|^2}$. Indeed, it is easy to check that $(T_s\Phi_0)(x) \equiv 0$. On the other hand, we have $(F\Phi_0)(x) = (F^{-1}\Phi_0)(x) = \Phi_0(x)$ (see [11, Lemma 7.6]). This implies $(T_c - iT_s)\Phi_0 = (T_c + iT_s)\Phi_0 = \Phi_0$. Since $T_c\Phi_0 = \Phi_0$.

Suppose that there exists an $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. By choosing $f(x) = \Phi_0(x)$ we have $\Phi_0 = \Phi_0 * e = e * \Phi_0$.

(i) Convolution (2.1). By the factorization identity of convolutions, we have $T_c\Phi_0 = T_c\Phi_0 T_c e.$ Since $T_c\Phi_0 = \Phi_0$, we get $\Phi_0 = \Phi_0 T_c e.$ By $\Phi_0(x) \neq 0$ for every $x \in \mathbb{R}^d$, we obtain $T_c e(x) = 1$ for every $x \in \mathbb{R}^d$. The last identity contradicts to the Riemann-Lebesgue as $\lim_{x\to\infty} T_c e(x) = 0$ (see [1, Theorem 31]).

(ii) Convolution (2.2). By the factorization identity $T_c\Phi_0 = (T_s\Phi_0)(T_s e).$

Using the above proved identity for Φ_0, we have $\Phi_0 = 0.$ This fails.

(iii) Convolutions (2.3), (2.4). By the factorization identities, we get $T_s\Phi_0 = T_c\Phi_0 T_s e.$ It follows $T_c\Phi_0 T_s e = 0.$ By $(T_c\Phi_0)(x) = \Phi_0(x) \neq 0$ for every $x \in \mathbb{R}^d$, we get $T_s e \equiv 0.$ Inserting this identity into the factorization identity we get that $T_s f = 0$ for every $f \in X$, which contradicts to $T_s \neq 0$ on X.

Hence, X has no unit.

We now prove the last conclusions of the theorem.

To end the proof we prove the non-commutativity of convolutions (2.1), (2.2), (2.3), (2.4). Suppose that one of them is commutative, i.e. $f * g = g * f$, for any $f, g \in X$. Changing variables $x - y := t, x + y := t$ in each of the integral terms in the left-side of the identities $f * g = g * f$, (for four convolutions), we find

$$\int_{\mathbb{R}^d} g(-x+u) f(u)du = \int_{\mathbb{R}^d} g(x+u) f(u)du, \text{ for almost every } x \in \mathbb{R}^d \qquad (3.1)$$

and for every $f, g \in L^1\mathbb{R}^d$ Write

$$\Omega := \{x \in \mathbb{R}^d : x_i \in [0,1], i = 1,...,d\}$$

the d-dimension box in $\mathbb{R}^d$. We set two functions $f, g \in L^1(\mathbb{R}^d)$ as follows

$$f(x) = \begin{cases} 1, & if\ x \in \Omega, \\ 0, & if\ x \notin \Omega, \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{2^d} x_1 ... x_d, & if\ x \in \Omega, \\ 0, & if\ x \notin \Omega. \end{cases}$$

By integrating, we get

$$\int_{\mathbb{R}^d} g(x+u) f(u)du = \begin{cases} \prod_{i=1}^d (x_i+1)^2, & x_i \in (-1,0], \forall i = 1,...,d, \\ \prod_{i=1}^d (1-x_i^2), & x_i \in (0,1], \forall i = 1,...,d, \\ 0, & \text{otherwise}, \end{cases}$$

$$\int_{\mathbb{R}^d} g(-x+u) f(u)du = \begin{cases} \prod_{i=1}^d (1-x_i^2), & x_i \in (-1,0], \forall i = 1,...,d, \\ \prod_{i=1}^d (-x_i+1)^2, & x_i \in (0,1], \forall i = 1,...,d, \\ 0, & \text{otherwise}, \end{cases}$$

The identity (3.1) fails in this case. The theorem is proved completely.

Remark 3.1. This theorem shows a fact that the convolution for one transform can be non-commutative. Namely, convolution (2.1) for T_c, T_c,T_c is noncommutative.

3.2 Integral equations of convolution type

Consider the convolution integral equation with the mixed Toeplitz-Hankel kernel

$$\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[k_1(x+y)+k_2(x-y)]\varphi(y)dy = p(x), \tag{3.2}$$

where $\lambda \in \mathbb{C}$ is predetermined, k_1, k_2, p are given, $\varphi(x)$ is to be determined.

Since the convolutions in Section 2 are considered in $L^1(\mathbb{R}^d)$ (not yet considered in $L^2(\mathbb{R}^d)$), given functions are assumed in $L^1(\mathbb{R}^d)$, and unknown function will be determined there. In what follows, the function identity $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbb{R}^d$. However, if the functions f, g are continuous, there should be emphasis that the identity $f(x) = g(x)$ is true for every $x \in \mathbb{R}^d$.

Now let us write

$$D_{T_c,T_s}(x) := \lambda^2 + 2\lambda T_c(k_2)(x) + T_c(k_1(x)+k_2(x))T_c(k_2(x)-k_1(x)) + T_s(k_1(x)+k_2(x))T_s(k_2(x)-k_1(x)), \tag{3.3}$$

$$D_{T_c}(x) := \lambda T_c p(x) + T_c(k_2(x)-k_1(x))T_c p(x) + T_s(k_2(x)-k_1(x))T_s p(x), \tag{3.4}$$

$$D_{T_s}(x) := \lambda T_s p(x) + T_c(k_1(x)+k_2(x))T_s p(x) - T_s(k_1(x)+k_2(x))T_c p(x). \tag{3.5}$$

Theorem 3.2. (a) *Assume that one the following conditions are fulfilled*:

(i) $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, and $\dfrac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}} \in L^1(\mathbb{R}^d)$.

(ii) $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$; and $\dfrac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}} \in L^1(\mathbb{R}^d)$.

If equation (3.2) *has solution in* $L^1(\mathbb{R}^d)$ *then it is solvable in a closed form:*

$$\varphi(x) = F^{-1}\left[\frac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}}\right](x), \quad \varphi(x) = F\left[\frac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}}\right](x) \tag{3.6}$$

according to conditions (i), (ii).

(b) Assume that conditions (i) *and* (ii) *are fulfilled. Then equation* (3.2) *has solution in* $L^1(\mathbb{R}^d)$ *if and only if*

$$F^{-1}\left[\frac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}}\right] = F\left[\frac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}}\right] \in L_1(\mathbb{R}^d). \tag{3.7}$$

Proof. We prove item (a). From convolutions (2.1), (2.2) it follows that

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+u)g(u)du = (f \underset{T_c}{*} g)(x) + (f \underset{T_c,T_s,T_s}{*} g)(x),$$

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-u)g(u)du = (f \underset{T_c}{*} g)(x) - (f \underset{T_c,T_s,T_s}{*} g)(x).$$

By the factorization identities of Theorems 2.1, 2.2 we get

$$T_c\left[\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+u)g(u)du\right] = T_c f(x)T_c g(x) + T_s f(x)T_s g(x), \tag{3.8}$$

$$T_c\left[\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-u)g(u)du\right] = T_c f(x)T_c g(x) - T_s f(x)T_s g(x), \tag{3.9}$$

for any $f, g \in L^1(\mathbb{R}^d)$.

Suppose that equation (3.2) has a solution $\varphi \in L_1(\mathbb{R}^d)$. Applying T_c to both sides of equation (3.2), using (3.8) and (3.9) we obtain

$$\lambda T_c\varphi(x) + T_c k_1(x) T_c\varphi(x) + T_s k_1(x) T_s\varphi(x) + T_c k_2(x) T_c\varphi(x) - T_s k_2(x) T_s\varphi(x) = T_c p(x). \quad (3.10)$$

On the other hand, from convolutions (2.3) and (2.4) it follows

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-u)g(u)du = (f \underset{T_s,T_s,T_c}{*} g)(x) + (f \underset{T_s,T_c,T_s}{*} g)(x),$$

$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+u)g(u)du = (f \underset{T_s,T_s,T_c}{*} g)(x) - (f \underset{T_s,T_c,T_s}{*} g)(x).$$

By the factorization identities of these convolutions we have

$$T_s\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x-u)g(u)du\right) = T_s f(x) T_c g(x) + T_c f(x) T_s g(x), \quad (3.11)$$

$$T_s\left(\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(x+u)g(u)du\right) = T_s f(x) T_c g(x) - T_c f(x) T_s g(x), \quad (3.12)$$

for any $f, g \in L^1(\mathbb{R}^d)$. Applying T_s to both sides of equation (3.2), using (3.11) and (3.12) we get

$$\lambda T_s\varphi(x) + T_s k_1(x) T_c\varphi(x) - T_c k_1(x) T_s\varphi(x) + T_s k_2(x) T_c\varphi(x) + T_c k_2(x) T_s\varphi(x) = T_s p(x). \quad (3.13)$$

Therefore, we have the system of two linear equations

$$\begin{cases} T_c\varphi(x)[\lambda + T_c(k_1+k_2)(x)] + T_s\varphi(x)[T_s(k_1-k_2)(x)] = T_c p(x), \\ T_c\varphi(x)[T_s(k_1+k_2)(x)] + T_s\varphi(x)[\lambda + T_c(k_2-k_1)(x)] = T_s p(x), \end{cases} \quad (3.14)$$

where $T_c\varphi(x)$, $T_s\varphi(x)$ are unknown functions. The determinants of system (3.14):
$D_{T_c,T_s}(x)$, $D_{T_c}(x)$, $D_{T_s}(x)$ as in (3.3), (3.4), (3.5).
Since $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, it is easy to find $(T_c\varphi)(x)$, $(T_s\varphi)(x)$.
Unfortunately, T_c and T_s have no inversion transforms. Now, we use the inversion transforms of the Fourier integral transform (see [11, Theorm 7.7]) to obtain function $\varphi(x)$ as follows.
Proof of conditions (i), (ii). Since $D_{T_c,T_s}(x) \neq 0$ for every $x \in \mathbb{R}^d$, we get

$$T_c\varphi(x) = \frac{D_{T_c}(x)}{D_{T_c,T_s}(x)}, \quad T_s\varphi(x) = \frac{D_{T_s}(x)}{D_{T_c,T_s}(x)}.$$

Hence

$$(F\varphi)(x) = \frac{D_{T_c}(x) - iD_{T_s}(x)}{D_{T_c,T_s}(x)}, \quad (F^{-1}\varphi)(x) = \frac{D_{T_c}(x) + iD_{T_s}(x)}{D_{T_c,T_s}(x)}.$$

Using the assumptions (i), (ii) and the inversion theorem of the Fourier integral transform, we get (3.6). Item (a) is proved. Now we prove item (b).
Necessity. Suppose that (3.2) has solution $\varphi \in L^1(\mathbb{R}^d)$. By the proof of item (a),

$$(F\varphi)(x) = \frac{D_{T_c}(x) - iD_{T_s}(x)}{D_{T_c,T_s}(x)}, \quad (F^{-1}\varphi)(x) = \frac{D_{T_c}(x) + iD_{T_s}(x)}{D_{T_c,T_s}(x)}.$$

Now we can apply the inversion theorem of the Fourier transform to obtain

$$\varphi(x) = F^{-1}\left(\frac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}}\right)(x), \text{ and } \varphi(x) = F\left(\frac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}}\right)(x).$$

The necessity is proved.

Sufficiency. Consider the function

$$\varphi(x) = F^{-1}\left[\frac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}}\right](x) = F\left[\frac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}}\right](x)$$

(this function belongs to $L^1(\mathbb{R}^d)$). By the inversion theorem of the Fourier trans form, we get

$$F\varphi = \frac{D_{T_c} - iD_{T_s}}{D_{T_c,T_s}}, \quad F^{-1}\varphi = \frac{D_{T_c} + iD_{T_s}}{D_{T_c,T_s}}.$$

Since $F = T_c - iT_s$ and $F^{-1} = T_c + iT_s$, we find two functions

$$T_c\varphi(x) = \frac{D_{T_c}(x)}{D_{T_c,T_s}(x)}, \quad T_s\varphi(x) = \frac{D_{T_s}(x)}{D_{T_c,T_s}(x)}, \qquad (3.15)$$

and they satisfy (3.14). Furthermore, we have

$$[\lambda + T_c(k_1 + k_2)(x)]T_c\varphi(x) = \frac{[\lambda + T_c(k_1 + k_2)(x)]D_{Tc}(x)}{D_{Tc,T_s}(x)},$$

$$T_s(k_1 - k_2)(x)T_s\varphi(x) = \frac{T_s(k_1 - k_2)(x)D_{Ts}(x)}{D_{Tc,T_s}(x)}.$$

Then

$$T_c\varphi(x)[\lambda + T_c(k_1 + k_2)(x)] + T_s\varphi(x)[T_s(k_1 - k_2)(x)] = T_c p(x).$$

Hence

$$T_c[\lambda\varphi + (k_1 + k_2) \underset{T_c}{*} \varphi + (k_1 - k_2) \underset{T_c,T_s,T_s}{*} \varphi(x) - p(x)] = 0.$$

By the similar procedure for the second function in we obtain

$$T_s[\lambda\varphi + (k_1 + k_2) \underset{T_s,T_s,T_c}{*} \varphi + (k_2 - k_1) \underset{T_s,T_c,T_s}{*} \varphi](x) = T_s p(x).$$

Using (3.8), (3.9), (3.10), (3.11), (3.12), (3.13) and $F = T_c - iT_s$ we get

$$F\left[\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[k_1(x + y) + k_2(x - y)]\varphi(y)dy\right] = Fp(x).$$

Hence,

$$F\left[\lambda\varphi(x) + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}[k_1(x + y) + k_2(x - y)]\varphi(y)dy - p(x)\right] = 0.$$

By the inversion theorem of the Fourier transform, the function $\varphi(x)$ satisfies equation (3.2) for almost every $x \in L^1(\mathbb{R}^d)$. The theorem is proved completely.

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