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AUTOISOMETRIES OF THE FOUR-DIMENSIONAL LIE ALGEBRA OF IV BIANCI TYPE

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1. Formulation of the problem. Let a Euclidean or Lorentz scalar product be introduced in the Lie algebra $\mathcal{G}$. Linear transformation $f:\mathcal{G}\to\mathcal{G}$ is called an autoisometry if it is both an isometry with respect to the scalar product and an automorphism of the Lie algebra. A transformation f is called an auto-similarity if it is both a similarity and an automorphism. As shown in [1], solving the problem of finding one-parameter self-similarity groups of a given Lie algebra allows us to construct self-similar homogeneous manifolds of the corresponding Lie group equipped with a left-invariant matrix. In the same paper, all the self-similarities of the three-dimensional Lie algebra Hs of the Heisenberg group were found for various ways of defining the Lorentzian scalar product on it, and the corresponding one-parameter similarity groups of the homogeneous Lorentzian manifold of the three-dimensional Heisenberg Lie group Hs were found.

The non-commutative two-dimensional Lie algebra $\mathcal{A}(1)$ is the Lie algebra of the group $A(1)$ of affine transformations of the line. The connected component of this Lie group containing the identity is the group $A^{+}(1)$ consisting of affine transformations preserving the orientation of the line. In [2] and [3], all ways of defining the Lorentz scalar product on two-dimensional and three-dimensional Lie algebras $\mathcal{A}(1)$ and $\mathcal{A}(1)\oplus\mathcal{R}$ were found, for which they admit one-parameter groups of auto-similarities and autoisometries, and formulas were written, on which these one-parameter groups act. This made it possible to construct self-similar connected manifolds of Lie groups $A^{+}(1)$ and $A^{+}(1)\times\boldsymbol{R}$. All possible one-parameter auto-similarity and auto-isometry groups for the four-dimensional Lie algebra $\mathcal{A}(1)\oplus\mathcal{R}^2$ were found in [4], and in [5] a similar problem was solved in the case of specifying a Euclidean scalar product in the Lie algebra.

The purpose of this work is to find all ways of defining the Lorentzian scalar product in the Lie algebra $\mathcal{G}_4=\mathcal{A}(1)\oplus\mathcal{A}(1)$ for which it admits a one-parameter autoisometry group, write out the matrix defining the action of this group, and prove that it does not admit self-similarities for any way of specifying the Lorentz scalar product on it. This four-dimensional Lie algebra $\mathcal{G}_4$ is the only one belonging to the IV type according to the Bianchi classification.

2. Lie algebra structure. In an appropriate basis (E_1,E_2,E_3,E_4) the commutation relations of the Lie algebra $\mathcal{G}_4$ are given by two equalities: $[E_3,E_1]=E_1, [E_4,E_2]=E_2$, and the remaining brackets are equal to the zero vector. We will call such a basis canonical. The two-dimensional subspace $\mathcal{H}$, which is the linear span of the vectors E_1 and E_2 is the derived Lie algebra $[\mathcal{G}_4,\mathcal{G}_4]$. This is a commutative ideal. The linear spans of the vectors E_1,E_3 and E_2,E_4 will be denoted by $\mathcal{L}_1$ and $\mathcal{L}_2$. These subspaces are two-dimensional non-commutative ideals.

The Lie algebra $\mathcal{G}_4$ admits a four-dimensional group of automorphisms, which it consists of transformations, which are given by matrices of the form

$$\begin{pmatrix}\alpha & 0 & \gamma & 0\\ 0 & \beta & 0 & \delta\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{1}$$

where $\alpha\neq 0, \beta\neq 0,$ and γ,δ can take any value. Such automorphisms will be called automorphisms of the first type. The simultaneous permutation of the basis vectors E_1 and E_2, E_3 and E_4 also preserves the bracket operation. It is given by a matrix of the form

$$\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}. \tag{2}$$

Therefore, the composition of automorphisms defined by matrices (1) and (2) is also an automorphism. It is given by the matrix

$$\begin{pmatrix} 0 & \alpha & 0 & \gamma \\ \beta & 0 & \delta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \text{ a }^1 0, \text{ b}^1 0. \qquad (3)$$

We will call them automorphisms of the second type. It is important to note that the determinant of matrices (1) and (3) is equal to $\alpha\beta$, i.e. is equal to the determinant of the matrix defining the restriction of the transformation to $\mathcal{H}$. Therefore, the transformation obtained by multiplying the identity transformation by a number is not an automorphism (we will call such transformations homotheties). A one-parameter group of auto-morphisms can contain only automorphisms of the first type for $\alpha > 0, \beta > 0$.

The linear span of vectors *x*, *y* will be denoted by $\langle x, y \rangle$.

3. Main results. Let the Lorentz scalar product be introduced on the Lie algebra $\mathcal{G}_4$.

Theorem. 1. *There is only one way to define the Lorentz scalar product in the Lie algebra* $\mathcal{G}_4 = \mathcal{A}(1) \oplus \mathcal{A}(1)$, *in which it admits a nontrivial one-parameter autoisometry group. The action of this group in the canonical basis is given by the matrix* (4) *and the Gram matrix of the basis has the form* (5).

$$\begin{pmatrix} e^{\nu t} & 0 & 0 & 0 \\ 0 & e^{-\nu t} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \nu > 0, t \in \boldsymbol{R}, \qquad (4) \qquad \Gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5)$$

2. *Lie algebra* $\mathcal{G}_4$ *does not admit autosimilarity for any way of specifying the Lorentz scalar product in it.*

Proof. Case 1. Lorentzian scalar product is induced on the ideal $\mathcal{H}$. Then a Euclidean scalar product is induced on the subspace $\mathcal{H}^{\perp}$. Let $h(t): \mathcal{G}_4 \to \mathcal{G}_4$ be a one-parameter group of autoisometries or autosimilarities. Under its action, the ideal $\mathcal{H}$, two isotropic directions in it, and one-dimensional ideals $\boldsymbol{R}E_1$, $\boldsymbol{R}E_2$ should remain invariant. In what follows, we denote $E_i' = h(t)(E_i), i = 1, 2, 3, 4.$

According to [6], in an appropriate basis (e_1, e_2, e_3, e_4), any one-parameter similarity group of a four-four-meter Minkowski space that has more than one invariant isotropic eigenvector is given by the matrix $e^{\mu t} F_1(t), \mu \geq 0,$ where

$$F_1(t) = \begin{pmatrix} e^{\nu t} & 0 & 0 & 0 \\ 0 & e^{-\nu t} & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{Q(t)}} \\ 0 & 0 & & \end{pmatrix}, \nu \geq 0, t \in \boldsymbol{R}, \qquad (6)$$

and $\mathbf{Q}(t)$ is an orthogonal matrix. In this case, the Gram matrix has the form (5).

In [7], all invariant two-dimensional subspaces of the transformation group were found, which is given by matrices of the form (6).

According to the theorem proved in [7] and the corollary from it holds $\mathcal{H} = \langle e_1, e_2 \rangle$, and we can conclude that $\mathbf{Q}(t) \equiv \mathbf{E}$. But it is important for us to clarify whether $h(t)$ is specified by the matrix $e^{\mu t} F_1(t)$ in the canonical basis, and also we need to find the Gram matrix of this basis. The determinant of the transformation matrix does not depend on the choice of the basis in the vector space. The determinant of the matrix $e^{\mu t} F_1(t)$ is equal to $e^{4\mu t}$, while the matrix of restriction of the transformation $h(t)$ to $\mathcal{H}$ has determinant equal to $e^{2\mu t}$. Therefore, $\mu = 0,$ and only one-parameter autoisometry group can exist.

If $\nu = 0,$ then the one-parameter group consists only of identical transformations. Let $\nu \neq 0$. Then the restriction of $h(t)$ to $\mathcal{H}$ cannot have more than two invariant directions. Therefore, E_1 and E_2 coincide with e_1 or e_2, i.e. they are isotropic (Figure 1).

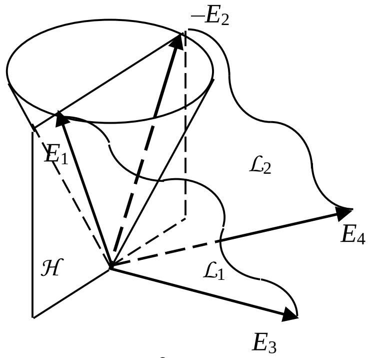

fig. 1

According to matrices (1) and (6), respectively

$$E_3'(t) = \gamma E_1 + E_3,\ E_2'(t) = e^{-\nu t} E_2.$$

Then

$$E_3'(t)\cdot E_2'(t) = \gamma e^{-\nu t} E_1\cdot E_2 + e^{-\nu t} E_3\cdot E_2 = \gamma e^{-\nu t} + e^{-\nu t} E_3\cdot E_2.$$

The last expression must be identically equal to $E_3 \cdot E_2$. From this we obtain the identity

$$\gamma e^{-\nu t} + (e^{-\nu t} - 1) E_3\cdot E_2 \equiv 0.$$

This is true only if $\gamma = 0$ and $E_3 \cdot E_2$ at the same time. Similarly, we get $\delta = 0$ and $E_4 \cdot E_1 = 0$. Since the scalar squares of the vectors E_1 and E_2 are equal to zero, then

$$E_3'(t)\cdot E_1'(t) = (\gamma E_1 + E_3)\cdot (e^{\nu t} E_1) = e^{\nu t} E_3\cdot E_1.$$

This expression must be equal to $E_3 \cdot E_1$. Therefore, $E_3 \cdot E_1 = 0$, and similarly we obtain $E_4 \cdot E_2 = 0$. It means that $\mathcal{H}^{\perp} = \langle E_3, E_4 \rangle$. Thus, the Gram matrix of the canonical basis has the form

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34} & g_{44} \end{pmatrix}, \quad \begin{vmatrix} g_{33} & g_{34} \\ g_{34} & g_{44} \end{vmatrix} > 0,\ g_{33} > 0.$$

Since $\gamma = \delta = 0$, we come to the conclusion that in the case under consideration there can only exist a one-parameter autoisometry group whose action in the canonical basis is given by the matrix (4).

Case 2. The Euclidean scalar product is induced on the ideal $\mathcal{H}$. Then Lorentzian scalar product is induced on the subspace $\mathcal{H}^{\perp}$, and a priori scalar squares of the vectors E_1 and E_2 are not equal to zero. Replace the basis vector E_3 by $\gamma E_1 + E_3$. Then

$$(\gamma E_1 + E_3)\cdot E_1 = \gamma E_1\cdot E_1 + E_3\cdot E_1$$

We will choose γ so that this expression becomes zero. In the same way, we choose δ so that the vector $\delta E_2 + E_4$ is orthogonal to E_2, and we replace the basis vectors E_3 and E_4 with $\gamma E_1 + E_3$ и $\delta E_2 + E_4$ respectively. Let us keep the previous notation for the new basis, which will also be canonical. Also, without changing the parenthesis operation, we can make the vectors E_1 and E_2 unit.

Let $h(t): \mathcal{G}_4 \to \mathcal{G}_4$ be a one-parameter group of autoisometries or autosimilarities. The isotropic directions of the subspace $\mathcal{H}^{\perp}$ must be invariant with respect to $h(t)$. Therefore, in respect to some basis (e_1, e_2, e_3, e_4) is given by the matrix $e^{\mu t} F_1(t)$. According to [7], we have $\mathcal{H} = \langle e_3, e_4 \rangle$, and thus, $\mathcal{H}^{\perp} = \langle e_1, e_2 \rangle$ (figure 2). From the condition on the determinant of the matrix of an automorphism, we again get $\mu = 0$ and only a one-parameter autoisometry group can exist.

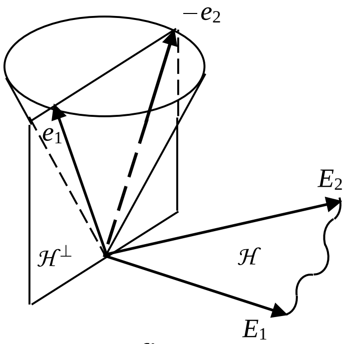

fig. 2

Moreover, the restriction of $h(t)$ to $\mathcal{H}$ has two invariant directions. This is possible only if the matrix $\mathbf{Q}(t)$ in (6) is identically equal to $\mathbf{E}$. We also need to prove that $\nu = 0$.

Suppose that $E_1 \cdot E_2 \neq 0$. We have

$$E_2'(t) = E_2, E_3'(t) = \gamma E_1 + E_3,$$

$$E_3'(t) \cdot E_2'(t) = \gamma E_1 \cdot E_2 + E_3 \cdot E_2 .$$

This expression must be identically equal to $E_3 \cdot E_2$. We get

$$\gamma E_1 \cdot E_2 \equiv 0 .$$

Hence $\gamma = 0$, and in a similar way we obtain $\delta = 0$, if $E_1 \cdot E_2 \neq 0$. It means that a one-parameter subgroup consists only of identical transformations. Therefore, we consider further only the case $E_1 \cdot E_2 = 0$. We have already proved that it is possible to choose a basis such that $E_3 \cdot E_1 = 0$. Therefore

$$E_3'(t) \cdot E_1'(t) = \gamma E_1 \cdot E_1 + E_3 \cdot E_1 = \gamma E_1 \cdot E_1 .$$

This expression must be identically equal to zero. Hence $\gamma = 0$, and in a similar way we obtain $\delta = 0$. So, in the case under consideration, there is no nontrivial one-parameter group of autoisometries and autosimilarities.

Case 3. A degenerate scalar product is induced on the subspace $\mathcal{H}$. Let $h(t): \mathcal{G}_4 \to \mathcal{G}_4$ be the one-parameter autoisometry or auto-similarity group. Then the only isotropic direction in $\mathcal{H}$ must be invariant under the action of the group. Let us first assume that it is the only invariant isotropic direction. Then, according to [6], with respect to some basis (e_1, e_2, e_3, e_4), the group $h(t)$ is given by the matrix $e^{\mu t} F_2(t), \mu \geq 0,$ where

$$F_2(t) = \begin{pmatrix} 1 & t & t^2/2 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

is the Gram matrix of the basis.

It was proved in [8] that in this case all invariant two-dimensional subspaces contain the vector e_1. Taking into account the structure of the Lie algebra, we come to the conclusion that $h(t)$ cannot be given by the matrix $e^{\mu t} F_2(t)$. Therefore, there is one more invariant isotropic direction, and in an appropriate basis (e_1, e_2, e_3, e_4) the group is given by the matrix $e^{\mu t} F_1(t)$. The subspace $\mathcal{H}$ can be invariant only if $\mathbf{Q}(t) \equiv \mathbf{E}$.

Suppose that $\nu \neq 0$. In this case, according to [7], the two-dimensional ideals $\mathcal{H}$, $\mathcal{L}_1$, and $\mathcal{L}_2$ are contained in the three-dimensional subspaces V_1 and V_2, which are given by the equations $x_1 = 0$ and $x_2 = 0$, respectively. These subspaces are orthogonal complements to e_1 and e_2, respectively, and have a common intersection $\langle e_3, e_4 \rangle$, so the ideal $\mathcal{H}$ can only be contained in one of them. Let it be V_1 (figure 3). Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are also contained in V_1, since they have a nonzero intersection with $\mathcal{H}$. This means that the linear span of $\mathcal{L}_1$ and $\mathcal{L}_2$ is three-dimensional. We got a contradiction, since the linear hull of $\mathcal{L}_1$ and $\mathcal{L}_2$ must be four-dimensional. Consequently, the one-parameter group $h(t)$ consists only of identical transformations.<

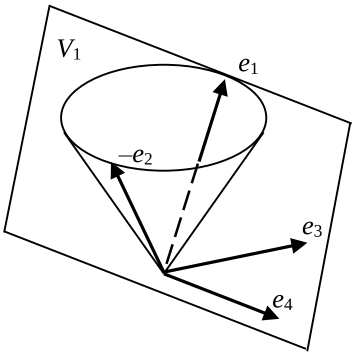

fig. 3

4. Conclusion. In this paper, we proved that four-dimensional Lie algebra $\mathcal{G}_4 = \mathcal{A}(1) \oplus \mathcal{A}(1)$ does not admit a one-parameter self-similarity group for any way of defining the Lorentz scalar product in it. Hence the corresponding connected Lie group $G_4 = A^+(1) \times A^+(1)$ equipped with a left-invariant Lorentzian metric cannot be a homogeneous self-similar Lorentzian manifold. The existing one-parameter autoisometry group for the algebra $\mathcal{G}_4$ allows us to construct in the future a one-parameter group of motions of the Lie group G_4, leaving the identity of the group fixed.

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