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THE SOLUTION OF THE ONE MULTIDIMENSIONAL INTEGRAL ABEL TYPE EQUATION WITH THE HYPERBOLIC SINE FUNCTION IN THE KERNEL OVER PYRAMIDAL DOMAIN

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Multidimensional integral equation of the first kind with the hyperbolic sine function in the kernel over pyramidal domain is studied. Solution of this equation in closed form is established, and necessary and sufficient conditions for its solvability in the space of summable functions are given.

Let us consider the following integral equation:

$$\frac{1}{\Gamma(\alpha)}\int\limits_{A_{c,r}(x)}\left(2\,\mathrm{sh}\frac{A\cdot(x-t)}{2}\right)^{\alpha-1} f(t)dt = g(x), \quad x\in A_{c,r}(b) \tag{1}$$

Given equation generalize the corresponding one-dimensional integral equation [1, §37.1].

Let $A=\|a_{jk}\|$ $\left(a_{jk}\in R^1\right)$ – be an $n\times n\,(n\in N)$ matrix with the determinant $|A|\neq 0$, we denote its vector-rows by $a_j=\left(a_{j1},...,a_{jn}\right)$ $(j=1,2,...,n)$, and the elements of the inverse matrix A^{-1} by a_{jk}. For vectors $\mathbf{x}=\left(x_1,x_2,...,x_n\right)\in R^n$ and $\mathbf{t}=\left(t_1,t_2,...,t_n\right)\in R^n$, $\mathbf{x}\cdot\mathbf{t}=\sum_{k=1}^{n}x_k t_k$ denotes their scalar product. For $\mathbf{x}\in R^n$ and $\alpha=(\alpha_1,...,\alpha_n)\in R^n$, $0<\alpha_i<1$ $(i=1,...,n)$ we set $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $\Gamma(\alpha)=\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)$. Let $A\cdot x=\left(a_1\cdot x,...,a_n\cdot x\right)$ and $(A\cdot x)^\alpha=(a_1\cdot x)^{\alpha_1}\cdots(a_n\cdot x)^{\alpha_n}$. For $\mathbf{c}=\left(c_1,c_2,...,c_n\right)\in R^n$ and $r\in R^1$ by $A_{c,r}(x)=\left\{t\in R^n: A\cdot(x-t)\geq 0,\ c\cdot t+r\geq 0\right\}$ – we denote the bounded *n* – pyramid in R^n. The expression $x\geq t$ means that $x_1\geq t_1,...,\ x_n\geq t_n$,; $dt=dt_1\cdots dt_n$; $f(t)=f(t_1,...,t_n)$. A function $\mathrm{sh}(x)$ such as: $\mathrm{sh}(x)=\prod_{j=1}^{n}\mathrm{sh}(x_j)$, where $\mathrm{sh}(x_j)$ $(j=1,2,...,n)$ – hyperbolic functions [1, §28.4; 2 – 4].

The solution of the equation (1) has the form:

$$f(x)=\prod_{k=1}^{n}\left(\sum_{j=1}^{n}a_{jk}\frac{\partial}{\partial x_j}\right)\left\{\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{-\alpha+1}{2}\right)}\int\limits_{A_{c,r}(x)}g(t)dt\right\}. \tag{2}$$

Consider the space [2-4]:

$$L_1\left(A_{c,r}(b)\right)=\left\{f(x):\int\limits_{A_{c,r}(\mathbf{x})}|f(t)|dt<\infty\right\};$$

$$I_{A_{c,r}}\left(L_1\right)=\left\{\varphi:\varphi(x)=\int\limits_{A_{c,r}(x),A\cdot(b-t)\geq A\cdot(x-t)}h(t)dt,\ h(t)\in L_1\left(A_{c,r}(b)\right)\right\}.$$

The space $I_{A_{c,r}}\left(L_1\right)$ plays the same role for the equation (1) as the space $AC([a,b])$ of absolutely continuous functions plays for the classical Abel integral equation [1, §2.2].

Theorem. *The multidimensional Abel-type integral equation (1)* $\alpha \in \mathrm{R}^n$ $(0 < \alpha < 1)$ *is solvable in the space* $L_1\left(A_{\mathbf{c},r}(\mathbf{b})\right)$ *if and only if:*

$$f_{A_{c,r}}^{\alpha}(\mathrm{x}) = f(\mathrm{x}) = \frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{-\alpha+1}{2}\right)} \int\limits_{A_{c,r}(\mathrm{x})} g(\mathrm{t})d\mathrm{t} \in I_{A_{c,r}}\left(L_1\right)$$

и $$\left[f_{A_{c,r}}^{\alpha}(\mathrm{x})\right]_{\mathrm{c}\cdot\mathrm{x}+r=0} = \left[\sum_{j=1}^{n} \tilde{a}_{jk} \frac{\partial}{\partial x_j} f_{A_{c,r}}^{\alpha}(\mathrm{x})\right]_{\mathrm{c}\cdot\mathrm{x}+r=0} = \ldots$$

$$\ldots = \left[\prod_{k=2}^{n}\sum_{j=1}^{n}\left(\tilde{a}_{jk}\frac{\partial}{\partial x_j}\right) f_{A_{\mathbf{c},r}}^{\alpha}(\mathrm{x})\right]_{\mathrm{c}\cdot\mathrm{x}+r=0} = 0 .$$

Under these conditions, the equation (1) is uniquely solvable in $L_1\left(A_{\mathbf{c},r}(\mathbf{b})\right)$ *and its solution is given by (2).*

REFERENCES

1. Самко, С.Г. Интегралы и производные дробного порядка и некоторые их приложения / С.Г. Самко, А.А. Килбас, О.И. Маричев. – Минск: Наука и техника, 1987. – 688с.
2. Kilbas, A.A. On integrable solution of a multidimensional Abel-type integral equation / A.A. Kilbas, M. Saigo, H. Takushima // Fukuoka Univ. Sci. Rep. – 1995. – Vol. 25. № 1. – P. 1 – 9.
3. Килбас, А.А. Решение многомерных гипергеометрических уравнений типа Абеля / А.А. Килбас, Р.К. Райна, М. Сайго, Г.М. Сривастава // Доклады НАН Беларуси. – 1995. – Т. 43. № 2. – С. 23 – 26.
4. Скоромник, О.В. Решение многомерного интегрального уравнения первого рода с функцией Куммера в ядре по пирамидальной области / О.В. Скоромник, С.А. Шлапаков // Веснік Віцебскага дзяржаўнага ўніверсітета. – 2014. – № 1. – С. 12–17.