UDC 512.7

CHARACTERISTIC AND MINIMAL POLYNOMIALS IN PROBLEMS

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In linear algebra, the minimal polynomial of an n-by-n matrix A over a field F is the monic polynomial p(x) over F of least degree such that p(A)=0. Any other polynomial q with q(A)=0 is a (polynomial) multiple of p. The following three statements are equivalent: λ∈F is a root of p(x), λ is a root of the characteristic polynomial of A, λ is an eigenvalue of A. The multiplicity of a root λ of p(x) is the geometric multiplicity of λ and is the size of the largest Jordan block corresponding to λ and the dimension of the corresponding Eigen space. The minimal polynomial is not always the same as the characteristic polynomial.

The minor of the kth order of a matrix of size (m×n), built on its rows with numbers$i_1 < i_2 < \dots < i_k \le m$ and columns with numbers$j_1 < j_2 < \dots < j_k \le n$ is called the main minor of the kth order, if$i_1 = j_1, i_2 = j_2, \dots, i_k = j_k$.Among all the main minors of a square matrix, its successive main diagonal minors are distinguished.

The sum of the diagonal elements of a square matrix A is called its trace and is denoted by *SpA*. The characteristic matrix of a square matrix $A(a_{ij})$ of the nth order is called the matrix $(A - \lambda E)$ with variable λ, taking any values.The determinant $|A - \lambda E|$is called the characteristic polynomial of the matrix A, and its roots λ_1, λ_2, …, λ_n– the characteristic roots or characteristic numbers of the matrix A.

Note that the characteristic polynomials of similar matrices are the same.

The characteristic polynomial of the matrix A of order n is a polynomial of the nth degree of λ and has the form:

$$|A - \lambda E| = (-1)^n(\lambda^n - p_1\lambda^{n-1} + p_2\lambda^{n-2} - \dots \pm p_n), \quad (1)$$

where p_m - the sum of the principal minors of the kth order of the matrix A, in particular, $p_1 = SpA$, $p_n = |A|$,*E* – unit matrix.

Let us consider the following problem of calculating the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix}.$$

By the definition of the characteristic polynomial, we get:

$$|A - \lambda E| = \begin{vmatrix} (1-\lambda) & 2 & 0 \\ 0 & (2-\lambda) & 0 \\ -2 & -2 & (-1-\lambda) \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2 = -(\lambda - 1)(\lambda + 1)(\lambda - 2).$$

Let us findp_1, p_2, p_3, using the formula (1):

$$p_1 = SpA = 2, \; p_2 = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix} = -1, \; p_3 = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{vmatrix} = -2.$$

So we have

$$|A - \lambda E| = (-1)^3(\lambda^3 - p_1\lambda^2 + p_2\lambda^1 - p_3) = (-1)^3(\lambda^3 - 2\lambda^2 - \lambda + 2) = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

Let us display the way of D.K. Fadeev to calculate the characteristic polynomial:

$$|A - \lambda E| = (-1)^n(\lambda^n - k_1\lambda^{n-1} - k_2\lambda^{n-2} - \dots - k_n). \quad (2)$$

This method consists in applying the following formulas:

$$\begin{gathered} A_1 = A,\ k_1 = SpA_1,\ B_1 = A_1 - k_1E, \\ A_2 = AB_1,\ k_2 = \tfrac{1}{2}SpA_2,\ B_2 = A_2 - k_2E, \\ A_{n-1} = AB_{n-2},\ k_{n-1} = \frac{1}{n-1}SpA_{n-1},\ B_{n-1} = A_{n-1} - k_{n-1}E, \\ A_n = AB_{n-1},\ k_n = \frac{1}{n}SpA_n,\ B_n = A_n - k_nE = 0. \end{gathered} \quad (3)$$

Equality $B_n = A_n - k_nE = 0$ is used to control computing.

The main problem will be solved by the Fadeev method. To solve the problem, we consistently find all the matrices included in formulas (3):

$$A_1 = A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix},\ k_1 = SpA_1 = 2,$$

$$B_1 = A_1 - k_1E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & -3 \end{pmatrix},$$

$$A_2 = AB_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{pmatrix},\ k_2 = \frac{1}{2}SpA_2 = 1,$$

$$B_2 = A_2 - k_2E = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 0 \\ 4 & -2 & 2 \end{pmatrix},$$

$$A_3 = AB_2 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 0 \\ 4 & -2 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix},\ k_3 = \frac{1}{3}SpA_3 = -2,$$

$$B_3 = A_3 - k_3E = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, according to the formula (2), we get:

$$|A - \lambda E| = (-1)^3(\lambda^3 - 2\lambda^2 - \lambda + 2) = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

If in an arbitrary polynomial from λ

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

instead of λ, put a square matrix*A* of order n, then matrix

$$P(A) = a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nE.$$

where*E* – unit matrix, is called the value of the polynomial P(A) at λ=A. If P(A)=0, than matrixA is called the matrix root of the polynomial, and P(λ) is called the polynomial, which is annulled by the matrix A.

Every square matrix A serves as the root of some nonzero polynomial.

The polynomial ϕ(λ) of the least degree with the highest coefficient equal to one, annulled by the matrix A, is called the minimum polynomial of this matrix.

Every polynomial P(λ), annulled by the matrix A, is exactly divisible by minimum polynomial ϕ(λ) of this matrix.

The characteristic polynomial $|A - \lambda E|$ of the matrix A and its minimum polynomial ϕ(λ) are related by the term

$$\varphi(\lambda) = \frac{(-1)^n \cdot |A - \lambda E|}{D_{n-1}}, \quad (4)$$

where D_{n-1}– the greatest common divisor of all (n-1)-order minors of the matrix $(A - \lambda E)$.

The roots of the minimal polynomial ϕ(λ) are all the different roots of the characteristic polynomial $|A - \lambda E|$.

Let's find the minimal polynomial for the matrix A. The characteristic polynomial for the matrix A it's:

$$|A - \lambda E| = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

The greatest common divisor D_2 of all second-order minors of the matrix

$$(A - \lambda E) = \begin{pmatrix} (1-\lambda) & 2 & 0 \\ 0 & (2-\lambda) & 0 \\ -2 & -2 & (-1-\lambda) \end{pmatrix}$$

is equalto one, since its minors

$$\begin{vmatrix} (1-\lambda) & 2 \\ -2 & -2 \end{vmatrix} = 2(\lambda + 1), \qquad \begin{vmatrix} 0 & (2-\lambda) \\ -2 & -2 \end{vmatrix} = 2(2 - \lambda)$$

are mutually simple. So,

$$\varphi(\lambda) = \frac{(-1)^3 \cdot |A - \lambda E|}{D_2} = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda + 1)(\lambda + 2).$$

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