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ABOUT ONE PROPERTY OF THE PRINCIPAL LEADING MINORS OF THE LOWER TRIANGULAR MATRIX

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In this paper, we considered a set of square matrices with positive angular minors. We implemented the similarity transformation triangular matrices with positive diagonal elements over the matrices. In Theorem 1, a sufficient condition was established to preserve the positivity of the angular minors of a square matrix with positive angular minors under the similarity transformation using the above-mentioned triangular matrices. In Theorem 2, a more rigorous sufficient condition was established for the preservation of a similar property with an additional restriction on the norm of the matrices under consideration.

Introduction. Analysis of numerical matrix characteristics in its various matrix transformations (e.g., in the similarity transformation, congruence, etc.) is one of the main problems in matrix theory. The aim of this paper is to study preservation of the principal leading corner minors' of a square n-dimensional matrix positivity and, moreover, of their separation from zero using the similarity transformation on the set of lower triangular matrices with positive diagonal elements.

Material and methods. We obtained the main results using the methods of linear algebra and matrix theory.

Findings and their discussion. Let $\mathbb{R}^n$ be a n-dimensional Euclidean vector space supplied with the norm $\|x\|=\sqrt{x^T x}$ (here the symbol T means the transpose of a matrix or a vector); $e_1, e_2, \ldots, e_n$ be vectors (columns) of the canonical orthonormal basis for the space $\mathbb{R}^n$; M_{mn} be the space of real $m\times n$-dimensional matrices supplied with the spectral (operator) norm $\|H\|=\max\limits_{\|x\|=1}\|Hx\|$, i.e. the norm induced by the Euclidean norm in the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ [1, p. 357]; $\mathrm{M}_n := \mathrm{M}_{nn}$. Denote by $E=[e_1,\ldots,e_n]\in \mathrm{M}_n$ the identity matrix. For any number $l\in\mathbb{N}$ denote the set of lower triangular $l\times l$-matrices with positive diagonal elements by $\mathcal{R}_l \subset \mathrm{M}_l$.

Definition 1. For any fixed number $k\in\{1,\ldots,n\}$ and any matrix $H=\{h_{ij}\}_{i,j=1}^n \in \mathrm{M}_n$ by $(H)_k\in \mathrm{M}_k$ denote it's *principal leading k-dimensional submatrix* [1, p. 30], i.e.

$$(H)_1=(h_{11}),\quad (H)_2=\begin{pmatrix} h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix},\quad \ldots,\quad (H)_n=H.$$

Determinants of leading principal submatrices of a matrix $H\in \mathrm{M}_n$ are called the *principal leading corner minors* [1, p. 30].

For any number $l\in\mathbb{N}$ by $\mathcal{H}_l\subset \mathrm{M}_l$ denote the set of l-dimensional matrices with positive principal leading minors, i.e.

$$\mathcal{H}_l := \{H\in \mathrm{M}_l:\ \det(H)_k>0,\ k=\overline{1,l}\}.$$

Definition 2. For any number $j=\overline{1,n}$ by $S_j\in \mathrm{M}_n$ denote the matrix received from a matrix R by changing its first j strings by the appropriate strings of a matrix H, i.e.

$$S_j := R+\sum_{i=1}^{j} e_i e_i^T (H-R),\quad j=\{1,\ldots,n\}.$$

In the sequel, with the use of monograph's terminology [2, p. 283], consider matrices $S_j \in \mathrm{M}_n$, $j = \overline{1,n}$, to be the *intermediate steps on the way from* R *to* H.

Definition 3. An ordered pair (R, H) of matrices on the set M_n is called *law-abiding* [2, p. 283], if the ratio $\det R > 0$ is correct and for each number $j \in \{1,\ldots,n\}$ of matrices S_j, which are the intermediate steps on the way from R to H, such ratios as $\det S_j > 0$ are correct.

Definition 4. *Square* n*-dimensional matrices* M and N are called *similar* [1, p. 61], if such matrix as $S\,(\det S \neq 0)$ exists so the following equality is correct

$$M = SNS^{-1}$$

and the transformation of matrix N itself with the use of the matrix S is called the similarity transformation.

Theorem 1. *Let* $R \in \mathcal{R}_n$, $H \in \mathcal{H}_n$. *The inclusion* $RHR^{-1} \in \mathcal{H}_n$ *is performed if and only if a pair* (R, H) *is law-abiding.*

The invariance of the positivity property of the principal leading corner minors of the matrix $H \in \mathcal{H}_n$ in the similarity transformation using the matrix $R \in \mathcal{R}_n$ is established by theorem 1, if these matrices are law-abiding. However, a stronger statement is also true. It is about the preservation of the principal corner minors' of the matrix H separation from zero in the similarity transformation using with not only the law-abiding condition but also the ρ -law-abiding of the same pair (R, H).

For any real numbers $r \geqslant 1$ and $\rho \in (0,1)$ let $\mathcal{R}_n(\rho, r) \subset \mathcal{R}_n$ be a set of lower triangular $n \times n$-matrices R with positive diagonal elements for which $\| R - E \| \leqslant r$ and $\det R \geqslant \rho$ are correct, i.e.

$$\mathcal{R}_n(\rho, r) := \{R \in \mathcal{R}_n : \| R - E \| \leqslant r, \det R \geqslant \rho\},$$

and a set of $n \times n$-matrices $\mathcal{H}_n(\rho, r) \subset \mathrm{M}_n$, for which $\| H - E \| \leqslant r$ and all the principal leading minors are not less than ρ, i.e.

$$\mathcal{H}_n(\rho, r) := \{H \in \mathrm{M}_n : \| H - E \| \leqslant r,\ \det(H)_k \geqslant \rho,\ k = \overline{1,n}\}.$$

Definition 5. Let $\rho \in (0,1)$ be a random fixed number. According to the definition from [2, p. 283] an *ordered pair* (R, H) on a set M_n^2 is *called* ρ *-law-abiding*, if $\det R \geqslant \rho$ and for any $j \in \{1,\ldots,n\}$ belonged to S_j, which are intermediate steps on the way from R to H, such inequalities as $\det S_j \geqslant \rho$ are correct.

Theorem 2. *Suppose* $r \geqslant 1$ *and* $\rho \in (0,1)$. *If a pair* (R, H) *for which* $R \in \mathcal{R}_n(\rho, r)$ *and* $H \in \mathcal{H}_n(\rho, r)$ *is* ρ *-law-abiding, then there exist such numbers as* $\rho_1 = \rho_1(\rho, r) \in (0,1)$ *and* $r_1 = r_1(\rho, r) \geqslant 1$, *for which* $RHR^{-1} \in \mathcal{H}_n(\rho_1, r_1)$.

Conclusion. The obtained results can be further used in the theory of controllability of asymptotic invariants of linear systems of ordinary differential equations in the study of global Lyapunov reducibility [2, p. 258 - 259] and even the global attainability [2, p. 253] of such systems.

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