UDC 517.983

# MULTY-DIMENTIONAL INTEGRAL TRANSFORM WITH THE CONFLUENT HYPERGEOMETRIC KUMMER FUNCTION IN THE KERNEL AND INTEGRAL EQUATION OF THE FIRST KIND

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*Multy-dimentional integral transform involving Kummer function in the kernel is studied on the space of summable functions on a finite domain* $[a_1,b_1]\times\ldots\times[a_n,b_n]\subset R^n$. *Mapping properties such as the boundedness are given, and the inversion formula is established. Integral equation of the first kind with the Kummer function in the kernel is also considered. The solution of the investigating equation is established, and conditions for it solvability in the space of summable functions are given.*

Consider the integral transformation on the left-hand side of (1)

$$\left(I_{\mathbf{a}+}^{\alpha,\beta,\lambda} f\right)(\mathbf{x}) \equiv \int_{a_1}^{x_1}\ldots\int_{a_n}^{x_n} \frac{(\mathbf{x}-\mathbf{t})^{\alpha-1}}{\Gamma(\alpha)} F\left(\beta;\alpha;\lambda(\mathbf{x}-\mathbf{t})\right) f(\mathbf{t})d\,\mathbf{t} = g(\mathbf{x}),\ \ \mathbf{x}>\mathbf{a}, \qquad (1)$$

here $\mathbf{x}=(x_1,\ldots,x_n)\in R^n$; $\mathbf{t}=(t_1,\ldots,t_n)\in R^n$ – vectors; $\mathbf{x}\cdot\mathbf{t}=\sum_{k=1}^{n} x_k t_k$; $\mathbf{x}>\mathbf{t}$ means $x_1>t_1,\ldots,\ x_n>t_n$; $\lambda=(\lambda_1,\ldots,\lambda_n)\in R^n$, $\lambda_i>0$, $(i=1,\ldots,n)$; $\beta=(\beta_1,\ldots,\beta_n)$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in R^n$, $0<\alpha_i<1$ $(i=1,\ldots,n)$; $\mathbf{a}=(a_1,\ldots,a_n)\in R^n$; $k=(k_1,\ldots,k_n)\in N^n$ – index with $k!=k_1!\cdots k_n!$ and $|k|=k_1+\ldots+k_n$; $D^k=\dfrac{\partial^{|k|}}{(\partial x_1)^{k_1}\cdots(\partial x_n)^{k_n}}$; $d\mathbf{t}=dt_1\cdots dt_n$; $(\mathbf{x}-\mathbf{t})^{\alpha-1}=(x_1-t_1)^{\alpha_1-1}\ldots(x_n-t_n)^{\alpha_n-1}$; $f(\mathbf{t})=f(t_1,\ldots,t_n)$. Function $F\left(\beta;\alpha;\lambda(\mathbf{x}-\mathbf{t})\right)=\prod_{j=1}^{n} {}_1F_1\left(\beta_j;\alpha_j;\lambda_j(x_j-t_j)\right)$, where ${}_1F_1(a;c;z)$ is the Kummer function [1, §1]:

$${}_1F_1(a;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!}=\lim_{b\to\infty} {}_2F_1\left(a,b;c;\frac{z}{b}\right),\ \ |z|<\infty, \qquad (2)$$

here $(a)_k$ – the Pochhammer symbol: $(a)_0\equiv 1$, $(a)_k=a(a+1)\ldots(a+k-1)$ $(a\in C; k\in N)$, ${}_2F_1(a,b;c;z)$ –Gauss hypergeometric function [1, §1]. The domain of integration of the transformation operator on the left-hand side of (1) is a rectangular parallelepiped with opposite vertices $\mathbf{x}=(x_1,\ldots,x_n)$ and $\mathbf{a}=(a_1,\ldots,a_2)$. In this paper, the transformation (1) is studied in space $L_{\bar{p}}(\mathbf{a},\mathbf{b})$, $[\mathbf{a},\mathbf{b}]=[a_1,b_1]\times\ldots\times[a_n,b_n]\subset R^n$, $-\infty<\mathbf{a}<\mathbf{b}<\infty$, $\bar{p}=(p_1,\ldots,p_n)$, $1\le\bar{p}<\infty$, functions $f(\mathbf{x})=f(x_1,\ldots,x_n)$ that have a finite norm [1, § 24.4]:

$$\|f\|_{\bar{p}}=\{\int_{a_n}^{b_n}[\ldots[\int_{a_2}^{b_2}[\int_{a_1}^{b_1}|f(x_1,\ldots,x_n)|^{p_1}dx_1]^{p_2/p_1}dx_2]^{p_3/p_2}\ldots]^{p_n/p_{n-1}}dx_n\}^{1/p_n}<\infty .$$

Expressions

$$\left(\mathrm{I}_{\mathbf{a}+}^{\alpha}\varphi\right)(\mathbf{x})=\frac{1}{\Gamma(\alpha)}\int_{a_1}^{x_1}\ldots\int_{a_n}^{x_n}\frac{\varphi(\mathbf{t})}{(\mathbf{x}-\mathbf{t})^{1-\alpha}}d\mathbf{t}\ (\alpha>0,\mathbf{x}>\mathbf{a}),$$

$$\left(D_{\mathbf{a}+}^{\alpha}f\right)(\mathbf{x})=\frac{1}{\Gamma(1-\alpha)}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\int_{a_1}^{x_1}\ldots\int_{a_n}^{x_n}\frac{f(\mathbf{t})d\mathbf{t}}{(\mathbf{x}-\mathbf{t})^{\alpha}}\quad (0<\alpha<1,\mathbf{x}>\mathbf{a})$$

are called respectively (left-sided) mixed fractional integral and mixed Riemann-Liouville fractional derivative of order $\alpha$ [1, § 24.2].

We introduce the space of functions

$$\mathrm{I}_{\mathbf{a}+}^{\alpha}(L_{\overline{p}}(\mathbf{a},\mathbf{b}))=\{f: f=\mathrm{I}_{\mathbf{a}+}^{\alpha}\varphi, \varphi\in L_{\overline{p}}(\mathbf{a},\mathbf{b}), \quad -\infty<\mathbf{a}<\mathbf{b}<\infty, 1<\overline{p}<\frac{1}{\alpha}, 0\le\alpha<1\}.$$

The space $\mathrm{I}_{\mathbf{a}+}^{\alpha}(L_{\overline{p}}(\mathbf{a},\mathbf{b}))$ plays the same role for equation (1) as the space $AC([a,b])$ of absolutely continuous functions for the classical Abel integral equation [1, §2.2].

Based on the representation of the kernel transform operator (1) through the series (2) and using the formula [1, formula (1.56)], we write the following formula, which reflects an operator structure (1):

$$I_{\mathbf{a}+}^{\alpha,\beta,\lambda}=I_{\mathbf{a}+}^{\alpha}\left(E-\lambda I_{\mathbf{a}+}^{1}\right)^{-\beta}, \tag{3}$$

where *E* is the identity operator.

On the basis of [1, Theorem 25.2, 37.1] we obtain.

**Theorem 1.** The operator on the left-hand side of (1) acts boundedly from the space $L_{\overline{p}}(\mathbf{a},\mathbf{b})$, $\overline{p}\ge 1$, onto the space $I_{\mathbf{a}+}^{\alpha}\left(L_{\overline{p}}(\mathbf{a},\mathbf{b})\right)$, $1<\overline{p}<\frac{1}{\alpha}$; $0\le\alpha<1$, $\lambda>0$, $-\infty<\mathbf{a}<\mathbf{b}<\infty$.

If in (1) $\mathbf{a}=0$ or $\mathbf{a}>0$, but the function is further defined by zero on the interval $0<\mathbf{t}<\mathbf{a}$, then we apply the multidimensional Laplace transform [1, formula (24.49)] to the left side of (1), we obtain

$$\left(L\, I_{\mathbf{0}+}^{\alpha,\beta,\lambda} f\right)(s)=s^{-\alpha}\left(1-\lambda s^{-1}\right)^{-\beta}\left(Lf\right)(s) \ \ (s=(s_1,\dots,s_n)>0; \lambda=(\lambda_1,\dots,\lambda_n)>0). \tag{4}$$

As formula (3) and (4) show, the Laplace transform $\left(Lh\right)(s)$ of the kernel $h(\mathbf{x})$ of operator (1) and the corresponding reconversion $\left[\left(Lh\right)(s)\right]^{-1}$ of it have the same form, differing only in the values of the parameters. In the simplest case of the fractional integration operator $I_{\mathbf{0}+}^{\alpha}$, which corresponds to the $\left(Lh\right)(s)=s^{-\alpha}$ with the condition $\mathrm{Re}(\alpha)>0$ [1, formula (24.50)], for the Laplace transform $s^{\alpha}$ of the kernel of the inverse operator $D_{\mathbf{0}+}^{\alpha}$, the condition $\mathrm{Re}(\alpha)>0$ forces us to represent $s^{\alpha}$ in the form $s^{\alpha}=s^{n}s^{-(n-\alpha)}$, where $\mathrm{Re}(n-\alpha)>0$, and the operator $\left(\frac{d}{d\mathbf{x}}\right)^{n}=D_{\mathbf{0}+}^{n}$ corresponds to the value $s^{n}$ [1, equality (18.12)]. Such operations we use for the conversion of the operator (1).

Using formula (4) and formula [2, formula 6.3 (7)], we obtain

$$f(\mathbf{x})=\left\{\left(I_{\mathbf{a}+}^{\alpha,\beta,\lambda}\right)^{-1}g\right\}(\mathbf{x})=I_{\mathbf{a}+}^{-\alpha}\left(E-\lambda I_{\mathbf{a}+}^{1}\right)^{\beta}g(\mathbf{x})=$$
$$=\frac{1}{\Gamma(1-\alpha)}\int_{\mathbf{a}}^{\mathbf{x}}\left(\mathbf{x}-\mathbf{t}\right)^{-\alpha}{}_1F_1\left(-\beta;1-\alpha;\lambda(\mathbf{x}-\mathbf{t})\right)\frac{dg(\mathbf{t})}{d\mathbf{t}}d\mathbf{t}. \tag{5}$$

**Theorem 2.** Consider the equation (1): $\left(I_{\mathbf{a}+}^{\alpha,\beta,\lambda}f\right)(\mathbf{x})=g(\mathbf{x})$. Let $0\le\mathbf{a}<\mathbf{x}<\mathbf{b}<\infty, 0<\alpha<1$, $\lambda>0$; function $g(\mathbf{x})$ is given on the $[\mathbf{a},\mathbf{b}]\subset R^{n}$; $f(\mathbf{x})$ - the required function (in the case $\mathbf{a}>0$ it is assumed that $f(\mathbf{x})=g(\mathbf{x})=0$ for $0<x<\mathbf{a}$), then the unique solution *f* of the equation (1) exists in the class of functions $L_{\overline{p}}(\mathbf{a},\mathbf{b})$, $\mathbf{b}<\infty$, if, $g(\mathbf{x})\in I_{\mathbf{a}+}^{\alpha}\left(L_{\overline{p}}(\mathbf{a},\mathbf{b})\right)$, $\overline{p}\ge 1$. In the case $\overline{p}=1$ the solution can be represented by the formula (5), if additional conditions $g(\mathbf{x})\in I_{\mathbf{a}+}^{\alpha}\left(L_{\overline{p}}(\mathbf{a},\mathbf{b})\right)$ and $g(\mathbf{a})=0$ are satisfied.

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