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ON SOME METHODS FOR STUDYING QUALITATIVE AND ASYMPTOTIC PROPERTIES OF SOLUTIONS TO HIGHER-ORDER QUASILINEAR DIFFERENTIAL EQUATIONS

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For the equation

$$y^{(n)} + \sum_{j=0}^{n-1} a_j(x) y^{(j)} = p(x)|y|^k \operatorname{sgn} y \ , \tag{1}$$

where $k > 1$, $n \geqslant 2$, the functions $p, a_0, \ldots, a_{n-1}$ are continuous for $x \geqslant 0$, we discuss some methods for studying qualitative and asymptotic properties of its solutions. (See, for example, [1–6]).

Theorem 1. *If the continuous functions* $a_0, \ldots, a_{n-1}$ *and* p *satisfy the conditions*

$$\int_{x_0}^{\infty} x^{n-j-1} |a_j(x)| \, dx < \infty \quad \text{for all} \quad j \in \{0, \ldots, n-1\} \tag{2}$$

and, for some integer number $m \in \{0, \ldots, n-1\}$, *the condition*

$$\int_{x_0}^{\infty} x^{n-1+(k-1)m} \, |p(x)| \, dx < \infty, \tag{3}$$

then for any $C \neq 0$ *there exists a solution* y *to equation (1) satisfying, as* $x \to \infty$,

$$y^{(j)}(x) \sim \frac{C\, m!\, x^{m-j}}{(m-j)!} \quad \text{for all } j \in \{0, \ldots, m\},$$

$$y^{(j)}(x) = o\left(x^{m-j}\right) \ \text{ and } \int^{\infty} s^{j-m-1} \left| y^{(j)}(s) \right| \, ds < \infty \ \text{ for all } j \in \{m+1, \ldots, n-1\}.$$

Sketch of the proof. To prove this theorem, we use a factorisation of the linear differential operator producing the left-hand side of (1). We use [7, Chap.1, Lemma 3.1, Lemma 3.2] and the following lemmas.

Lemma 1. *If continuous functions a_j satisfy inequalities (2), then for any $h \neq 0$ the equation*

$$y^{(n)}(x) + \sum_{j=0}^{n-1} a_j(x) y^{(j)}(x) = 0$$

has a $\mathcal{C}^n$ solution $y(x)$ such that

$$\begin{aligned} &y(x) \to h && \text{as } x \to \infty, \\ &x^j y^{(j)}(x) \to 0 && \text{as } x \to \infty, \quad j = 1, \dots, n-1, \end{aligned}$$

$$\int_0^\infty x^{j-1} \left| y^{(j)}(x) \right| dx < \infty, \quad j = 1, \dots, n.$$

Lemma 2. *Any linear differential operator*

$$L = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} a_j(x) \frac{d^j}{dx^j},$$

where all continuous functions a_j satisfy (2), can be represented in a neighborhood of $+\infty$ as the composition operator

$$L = b_0 B_1 \circ \cdots \circ B_n,$$

where all B_j, $j = 1, \dots, n$, are the first-order operators $u \mapsto \dfrac{d}{dx}(b_j u)$ and each b_j, $j = 0, \dots, n$, is a $\mathcal{C}^j$ function satisfying at infinity

(i) $b_j(x) \to 1$,

(ii) $x^i b_j^{(i)}(x) \to 0$ *for all* $i \in \{1, \dots, j-1\}$,

(iii) $\int\limits_{x_0}^{\infty} x^{i-1} \left| b_j^{(i)}(x) \right| dx < \infty$ *for all* $i \in \{1, \dots, j\}$ *and some* $x_0 \in \mathbb{R}$.

Remark 1. Note, that the first statement of this theorem even in a more general case ($k > 0$ instead of $k > 1$) follows from [4, Corollary 8.2] obtained by a quite different method.

Remark 2. In [6], under conditions (2) and more strong than (3) condition

$$\int_0^\infty x^{(n-1)(k+1)} |p(x)| dx < \infty,$$

a more strong result is obtained: it is proved that for any $C_1, \dots, C_{n-1}$ there exists a solution $y(x)$ to equation (1) such that

$$u(x) = \sum_{j=0}^{n-1} C_j \xi_j(x) + o(1), \quad x \to \infty,$$

where the functions ξ_j form a fundamental system of solutions to equation (1) with $p = 0$, and

$$\xi_j = \frac{x^j}{j!}(1 + o(1)), \quad x \to \infty.$$

Remark 3. In [5], by the same method of a suitable representation for the linear differential operator, under some conditions on a_j, $j = 0, \ldots, n-1$, a criterion is obtained for equation (1) to have a solution equivalent at infinity to any non-zero constant, and an oscillatory criterion was proved.

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A FORMULA FOR THE BOHL EXPONENT OF DISCRETE TIME-VARYING SYSTEMS

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It is known that the stability of linear time-varying systems is not determined by the position of the spectra of the coefficient matrices. There are examples of continuous-time systems, the coefficients of which have spectra lying in the left half-plane, and these systems are not stable, and conversely all matrices of a stable system may have spectra of the coefficients lying in the right half-plane (see e.g. [6], p. 257). Similarly, one can give examples of stable, asymptotically stable, and even uniformly exponentially stable discrete-time systems with time-varying coefficients, whose coefficient matrices have eigenvalues outside the unit circle, as well as unstable systems with matrices which all eigenvalues lying inside the unit circle. However, if the coefficient matrices of a discrete time-varying system have spectra in the unit circle, stability can be guaranteed by a sufficiently slow variation of the coefficients. This is the basic idea behind the so-called freezing method that was, for discrete-time systems, for the first time described in [2]. A comprehensive description of the results obtained with this technique is provided in Section 10.1 of [4] (see also [3], [5] and the references therein).

Uniform exponential stability of a linear system is characterized by the Bohl exponent. A system is uniformly asymptotically stable if and only if the Bohl exponent is negative ([6], Theorem 3.3.15). The above-mentioned lack of dependence between the spectra of the coefficients and the stability causes that in general it is not possible to give the formula for the Bohl exponent expressed by the eigenvalues of the coefficients. However, as noted by V. M. Millionschikov in [8] and by J. Daleckii and M.G. Krein in the monograph [1, p. 200], such a formula can be given for continuous-time systems with weak variation by Persidskii (see also [7], Section 3.6). The main result of this note is to provide such a formula for discrete systems. On the basis of this formula, we will obtain the necessary and sufficient