CONSTRUCTING A "BENDIXSON'S BAG" FOR A DYNAMICAL SYSTEM USING DN-TRACKING METHOD

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One of the main problems of the qualitative theory of dynamical systems is to prove the existence of a closed trajectory [1-5]. To prove its existence even for a concrete two-dimensional dynamical system remains a difficult task. In 2008, A.Azamov proposed a new method called DN-tracking of trajectories of dynamical system, which under certain conditions allows one to rigorously prove the specific properties of dynamical systems (namely, existence of closed trajectory) based on numerical methods [4].

Consider the simplest non-linear dynamical system

$$\begin{cases} \dot{x} = ax + y + x^2, \\ \dot{y} = bx + y, \end{cases} \tag{1}$$

where a and b are real parameters.

This system can be written in a vector form as

$$\dot{z} = f(z), \tag{2}$$

where $z = (x, y)$, $f(z) = (ax + y + x^2,\, bx + y)$.

It is known that almost many dynamical systems are not integrable in explicit way and one can not find its first integral forms, so to find approximate solution of system (2) with given initial condition $z(0) = z_0$ one has to use numerical methods. In spite of the simplicity of system (2) it is not integrable and we call it the simplest model system on the plane.

To solve it numerically we applied an explicit Runge-Kutta method of second order:

$$z_{n+1} = z_n + F(z_n,\ h), \tag{3}$$

where $z_n = (x_n,\ y_n)$, $F(z_n, h) = hf\left(z_n + \dfrac{h}{2}f(z_n)\right)$.

Further an approximate solution z_n obtained by above scheme (3) we call the discrete trajectory of the system (2). Note that one cannot find exact values of discrete solution in spite of the fact that the system (2) is polynomial. Therefore, one has to work with another sequence of vectors ζ_n to be obtained by rounding the values of z_n on a computer. Indeed, in real calculations, due to rounding of the results of arithmetic operations on a computer, instead of the sequence z_n one get another sequence ζ_n, which we call it a numerical trajectory of the system (2) that satisfies the following a recursion formula

$$\zeta_{n+1} = \zeta_n + \tilde{F}(\zeta_n,\ h), \tag{4}$$

where $\tilde{F}(\zeta_n, h)$ is the approximate value of $F(\zeta_n,\ h)$ which is calculated on a computer.

It should be noted that one can get the following estimate

$$\left|\tilde{F}(\zeta_n,\ h) - F(\zeta_n,\ h)\right| < \Delta, \tag{4a}$$

where Δ is upper bound of round-off error that produced on a computer after calculation the value of $F(\zeta_n, h)$.

Computer result shows that system (2) has a closed trajectory $z(t)$ of the period $T \in (8.05,\ 8.15)$ passing through near the point $z_0 = (0.6 \pm 0.01,\ 0)$ when $a = -0.8$

and $b=-2.$ For other values of parameters a, b one can experimentally establish existence of closed trajectory on a computer.

Let $m_0 = \max_{z\in K} \|f(z)\|$, $m_1 = \max_{z\in K} \left\|\frac{\partial f(z)}{\partial z}\right\|$, $m_2 = \max_{z\in K} \left\|\frac{\partial^2 f(z)}{\partial z^2}\right\|$, $h=\frac{T}{N}$, N is a number of division of given interval $[0,\ T]$.

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Assumption 1. *The trajectory $z(t)$ with given initial condition $z(0)=z_0$ exists on the time interval $0\leqslant t\leqslant T$ and $z(t)\in K$.*

Theorem 1. *Assumption 1 holds on time interval* $\left[0,\ \dfrac{T}{2}\right]$ *and*

$$|z(t)-\zeta_n| < 9.2\cdot 10^{-4} = \varepsilon,$$

where $n=\left[\dfrac{t}{h}+\dfrac{1}{2}\right]$.

The last inequality means that it possible to trace with accuracy ε the trajectory $z(t)$ by means of a sequence of vectors ζ_n, actually computable and stored in the memory of the computing device. Note that in worse case, exact trajectory $z(t)$ may pass on the leftmost point C of the line segment $C'C$, where $C'=\zeta_N+(\varepsilon,\ 0)$, $C=\zeta_N-(\varepsilon,\ 0)$. Therefore, in the next step, we study exact trajectory with initial condition $z_0=C$ on second segment

$$\left[\frac{T}{2},\ T\right].$$

Since our system (2) is dynamical systems, instead of the segment $\left[\dfrac{T}{2}, T\right]$ one may consider the segment $\left[0,\ \dfrac{T}{2}\right]=[0,\ 4.136]$.

Therefore we proved that the trajectory $z(t)$ starting from the initial point A returns to the left side that point on time interval $[0, T]$ and intersects OX-axis at some point I which placed on the left of point A. So the region B which is bounded by the inner part of the union the lines both trajectory $z(t)$ and line segment IA serves as a "Bendixson's bag".

Refrences

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