We have from (4)

$$\|X\|_{\mathbb{C}} \leqslant \frac{\|X_1\|_{\mathbb{C}}}{1-q}. \tag{5}$$

From (3) for $X_0 = 0$ we obtain an estimate for X_1:

$$\|X_1\|_{\mathbb{C}} \leqslant \gamma\mu\omega\varepsilon h. \tag{6}$$

Using (6) and (5) we have

$$\|X(t)\|_{\mathbb{C}} \leqslant \frac{\gamma\mu\omega\varepsilon h}{1-q(\rho)}.$$

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ADDITIONAL DARBOUX POLYNOMIALS OF HAMILTONIAN SYSTEMS

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Consider a canonical polynomial Hamiltonian system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \qquad \frac{dp_i}{dt} = -\,\partial_{q_i} H(q,p), \quad i = 1, \ldots, n, \tag{1}$$

where $q = (q_1, \ldots, q_n) \in \mathbb{C}^n$ and $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$ are the generalized coordinates and momenta, respectively, the independent variable $t \in \mathbb{R}$, and the Hamiltonian function $H\colon \mathbb{C}^{2n} \to \mathbb{C}$ is a polynomial of degree $h \geqslant 2$. In this paper, we use the Darboux theory of integrability (or the theory of partial integrals) [1, 2] for studying the existence of additional Darboux polynomials (or partial integrals) for the polynomial Hamiltonian system (1) in the phase space $\mathbb{C}^{2n}$.

The Darboux theory was established [1] by the French mathematician J. G. Darboux and provided a link between the existence of first integrals and Darboux polynomials for polynomial autonomous differential systems. Notice that for the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see, for example, the monograph [2] and the references therein). In the articles [3 – 5], we applied the Darboux theory of integrability for the polynomial Hamiltonian systems (1). This paper continues studies in this direction and develops approaches of construction additional Darboux polynomials by known Darboux polynomials. For example, the following statement holds.

Theorem. *Suppose the polynomial Hamiltonian system (1) has a Darboux polynomial* $w \in \mathbb{C}[q,p] \setminus \mathbb{C}$ *such that the Poisson bracket*

$$\big[w(q,p), H(q,p)\big] = w(q,p)\, M(q,p) \quad \text{for all } (q,p) \in \mathbb{C}^{2n},$$

where the polynomial $M \in \mathbb{C}[q,p]$ *such that* $\deg M \leqslant h-2$. *And let a map*

$$\psi = (\psi_1(q,p), \ldots, \psi_n(q,p)), \quad \varphi = (\varphi_1(q,p), \ldots, \varphi_n(q,p)) \ \ \textit{for all } (q,p) \in \mathbb{C}^{2n}$$

be holomorphism of $\mathbb{C}^{2n}$ *to* $\mathbb{C}^{2n}$ *such that the following identities hold*

$$\big[\psi_i(q,p), H(q,p)\big] = \lambda \partial_{\varphi_i} H(\psi,\varphi)\ \Big|_{\substack{\psi = \psi(q,p),\\ \varphi = \varphi(q,p)}}, \quad \big[\varphi_i(q,p), H(q,p)\big] = -\lambda \partial_{\psi_i} H(\psi,\varphi)\ \Big|_{\substack{\psi = \psi(q,p),\\ \varphi = \varphi(q,p)}},$$

where a number $\lambda \in \mathbb{C} \setminus \{0\}$. *Then the polynomial*

$$\widetilde{w}(q,p) = \ w(\psi(q,p), \varphi(q,p)) \quad \textit{for all } (q,p) \in \mathbb{C}^{2n}$$

is an additional Darboux polynomial of the Hamiltonian system (1) such that the Poisson brackets

$$\big[\widetilde{w}(q,p), H(q,p)\big] = \lambda\, \widetilde{w}(q,p)\, M(\psi(q,p), \varphi(q,p)) \quad \text{for all } (q,p) \in \mathbb{C}^{2n}.$$

Aknowledgement. Research was supported by Horizon2020-2017-RISE-777911 project.

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