$$Q_{11}(t) = -\frac{1}{\tau}\Big(\frac{d}{dt}H(t) + H(t)A(t) + A^*(t)H(t) + M(0,t)\Big) - K(0),$$

$$Q_{12}(t) = \frac{1}{\tau}\big(H(t)A(t) + M(0,t)\big)D(t) + K(0)D(t), \quad Q_{22}(t) = R(t),$$

$$Q_{13}(t,s) = -H(t)B(t,s), \quad Q_{33}(s) = K(s).$$

Теорема. *Пусть существуют T-периодическая гладкая матрица $H(t) = H(t)^* > 0$ и матрицы $K(s) = K^*(s) > 0$, $\frac{d}{ds}K(s) < 0$, $s \in [0,\tau]$, и $M(s,\xi) = M^*(s,\xi) > 0$, $\frac{\partial}{\partial s}M(s,\xi) < 0$, $s \in [0,\tau]$, $\xi \in \mathbb{R}$, такие, что*

$$R(t) > 0, \quad t \in [0,T], \quad \int\limits_0^T \gamma_H(s)ds > 0,$$

где $\gamma_H(t) = \min\left\{p^H_{min}(t), k\right\}$, $k > 0$ – максимальное число такое, что

$$\frac{d}{ds}K(s) + kK(s) \leqslant 0, \quad \frac{\partial}{\partial s}M(s,\xi) + kM(s,\xi) \leqslant 0, \quad s \in [0,\tau], \quad \xi \in \mathbb{R}.$$

Тогда нулевое решение системы экспоненциально устойчиво.

Отметим, что помимо представленных достаточных условий экспоненциальной устойчивости получены конструктивные оценки решений системы (1), которые характеризуют экспоненциальное убывание решений на бесконечности.

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ON THE ROBUST STABILIZABILITY ANALYSIS OF THREE-TIME-SCALE LINEAR TIME-INVARIANT SINGULARLY PERTURBED SYSTEMS WITH DELAY

C.A. Naligama, O.B. Tsekhan

Let $p \triangleq \frac{d}{dt}$ be the differentiation operator, $h = \text{const} > 0$, e^{-ph} be the delay operator: $e^{-ph}v(t) = v(t-h)$, $e^{-jph}v(t) = v(t-jh)$. The following Three-time-scale Singularly Perturbed Linear Time-invariant System with Multiple Commensurate Delays in the slow state variables (TSPLTISD) is considered in the matrix-operator form:

$$\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{bmatrix} = A\left(\varepsilon_1, \varepsilon_2, e^{-ph}\right)\begin{bmatrix}x(t)\\ y(t)\\ z(t)\end{bmatrix} + B\left(\varepsilon_1, \varepsilon_2\right)u(t), \ x \in \mathbb{R}^{n_1}, \ y \in \mathbb{R}^{n_2}, \ z \in \mathbb{R}^{n_3}, \quad (1)$$

with initial conditions: $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$, $x(\theta) = \varphi(\theta)$, $\theta \in [-h, 0)$.

Here,

$$A\left(\varepsilon_1,\varepsilon_2,e^{-ph}\right)=\begin{bmatrix} A_{11}\left(e^{-ph}\right) & A_{12} & A_{13} \\ \dfrac{A_{21}\left(e^{-ph}\right)}{\varepsilon_1} & \dfrac{A_{22}}{\varepsilon_1} & \dfrac{A_{23}}{\varepsilon_1} \\ \dfrac{A_{31}\left(e^{-ph}\right)}{\varepsilon_2} & \dfrac{A_{32}}{\varepsilon_2} & \dfrac{A_{33}}{\varepsilon_2} \end{bmatrix},\quad B\left(\varepsilon_1,\varepsilon_2\right)=\begin{bmatrix} B_1 \\ \dfrac{B_2}{\varepsilon_1} \\ \dfrac{B_3}{\varepsilon_2} \end{bmatrix}, \tag{2}$$

$$A_{i1}\left(e^{-ph}\right)\triangleq\sum_{j=0}^{l}A_{i1j}e^{-jph},\quad i=\overline{1,3},$$

be matrix operators, $A_{i1j}, A_{i2}, A_{i3}, B_i,\ i=1,2,3,\ j=\overline{0,l}$, be constant matrices of appropriate dimensions; $0<\varepsilon_2\ll\varepsilon_1\ll 1$; $x,\ y,\ z$ are the slow, fast and fastest variables, respectively; $x_0\in\mathbb{R}^{n_1},\ y_0\in\mathbb{R}^{n_2},\ z_0\in\mathbb{R}^{n_3},\ \varphi(\theta),\ \theta\in[-h,0)$, is a piecewise continuous n_1-vector function; $u\in U,\ U$ is a set of piecewise continuous r-vector functions for $t\geqslant 0$.

Definition. For a given $\varepsilon_1>0,\ \varepsilon_2>0$ TSPLTISD (1) is considered *to be stabilizable* if there exists the linear feedback controller,

$$u(t)=\begin{pmatrix}F_1(e^{-ph}) & F_2 & F_3\end{pmatrix}\begin{bmatrix}x(t)\\ y(t)\\ z(t)\end{bmatrix}, \tag{3}$$

with, $F_1(e^{-ph})\in\mathbb{R}^{r\times n_1},\ F_2\in\mathbb{R}^{r\times n_2}$ and $F_3\in\mathbb{R}^{r\times n_3}$, such that the closed-loop system (1), (2) is exponentially stable for the given, $\varepsilon_1>0,\ \varepsilon_2>0$ and $t\geqslant 0$.

If there exist numbers, $\varepsilon_1^*>0,\ \varepsilon_2^*>0$; for which TSPLTISD (1) is stabilizable for any $\varepsilon_1\in(0,\varepsilon_1^*]$ and $\varepsilon_2\in(0,\varepsilon_2^*]$, the complete stabilizability is robust with respect to the parameters $\varepsilon_1,\ \varepsilon_2$.

By $\mathbb{C}$, the set of complex numbers is denoted. Let's consider $S(\mathbb{C})$ to be in the set of all the complex numbers $\mathbb{C}$ with negative real parts: $S(\mathbb{C})=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$.

Applying for a given $\varepsilon_1>0,\ \varepsilon_2>0$ to the TSPLTISD (1) the criterion [1] (Theorem 2.) for the stabilizabilty of the state delay system, we obtain:

Theorem 1. *For a given $\varepsilon_1>0$ and $\varepsilon_2>0$ the system (1) is stabilizable if and only if:*

$$\operatorname{rank}\left[\lambda I_{n_1+n_2+n_3}-A(\varepsilon_1,\varepsilon_2,e^{-\lambda h}),B\left(\varepsilon_1,\varepsilon_2\right)\right]=n_1+n_2+n_3\quad\forall\lambda\in\mathbb{C}\setminus S(\mathbb{C}). \tag{4}$$

Assumptions. $\det\ A_{33}\neq 0$ and $\det\ [A_{22}-A_{23}A_{33}^{-1}A_{32}]\neq 0$.

Under the Assumptions the following n_1-dimensional degenerate system with delay (slow subsystem, DS), the n_2-dimensional ε_1-Boundary Layer System (ε_1-BLS) without delay and the n_3-dimensional ε_2-Boundary Layer System (ε_2-BLS) without delay can be obtained:

$$\dot{x}_s(t)=A_s\left(e^{-ph}\right)x(t)+B_su(t),\quad x_s\in\mathbb{R}^{n_1}, \tag{5}$$

$$\frac{d\hat{y}(\tau_{\varepsilon_1})}{d\tau_{\varepsilon_1}}=A_{f_{\varepsilon_1}}\hat{y}(\tau_{\varepsilon_1})+B_{f_{\varepsilon_1}}u_{f_{\varepsilon_1}}(\tau_{\varepsilon_1}),\quad \hat{y}\in\mathbb{R}^{n_2}, \tag{6}$$

$$\frac{d\hat{z}(\tau_{\varepsilon_2})}{d\tau_{\varepsilon_2}}=A_{f_{\varepsilon_2}}\hat{z}(\tau_{\varepsilon_2})+B_{f_{\varepsilon_2}}u_{f_{\varepsilon_2}}(\tau_{\varepsilon_2}),\quad \hat{z}\in\mathbb{R}^{n_3}. \tag{7}$$

Theorem 2. *The Degenerate System (5) of the TSPLTISD (1) is stabilizable if and only if:*

$$\operatorname{rank}\left[\lambda I_{n_1}-A_s(e^{-\lambda h}),B_s\right]=n_1,\quad\lambda\in\mathbb{C}\setminus\mathbb{S}(\mathbb{C}). \tag{8}$$

Similarly, the Theorems on the stabilizabilty of the ε_1-BLS(6), ε_2-BLS(7) are established.

Theorem 3. *Let the Assumptions be satisfied, the DS (4), the ε_2-BLS (6) and the ε_1-BLS (7) be stabilizable by a linear state feedback controls $F_s(e^{-ph})$, F_{ε_2}, F_{ε_1}, respectively, conditions*

$$\det\,(A_{33} + B_3F_3) \neq 0,$$

$$\det\,[(A_{22} + B_2F_2) - (A_{23} + B_2F_3)(A_{33} + B_3F_3)^{-1}(A_{32} + B_3F_2)] \neq 0, \tag{9}$$

where the matrix operator $F_1(e^{-ph})$ and matrices F_2 and F_3 are expressed in terms of the matrix parameters of the system (1) and $F_s(e^{-ph})$, F_{ε_2}, F_{ε_1}, are satisfied. Then, there exist ${\varepsilon_1}^* > 0$ and ${\varepsilon_2}^* > 0$ such that the TSPLTISD (1) is stabilizable for all $\varepsilon_1 \in (0, \varepsilon_1^*]$ and $\varepsilon_2 \in (0, \varepsilon_2^*]$, (i.e. robust with respect to the small parameters ε_1, ε_2) by a controller in the composite form (3).

Sketch of Proof. The proof follows [3,4] using the results from [2,5]. Consider a feedback law of the following form $u(t) = u_s(t) + u_1(\tau_{\varepsilon_1}) + u_2(\tau_{\varepsilon_2})$ where

$$u_s(t) = F_s(e^{-ph})x_s(t), \quad u_1(t) = F_{\varepsilon_1}\tilde{y}(\tau_{\varepsilon_1}), \quad u_2(t) = F_{\varepsilon_2}\tilde{z}(\tau_{\varepsilon_2}).$$

Then by replacing x_s by x, $\tilde{y}$ by $y - y_s$, $\tilde{z}$ by $z - z_s$, we obtain the control law of the form (3). Applying the feedback law (3) to TSPLTISD (1) we obtain a closed-loop system of the form of TSPLTISD (1). Since (9), then for sufficient small parameters the closed-loop system can be decomposed into the exact slow, fast and fastest subsystems by the appropriate nonsingular transformation [2, 5]. Then it is proved that the decoupled closed-loop system is asymptotically close to closed-loop subsystems: DS and BLSs, which, according to the assumption of the theorem, are stabilizable. Considering the preservation of the full rank of a matrix under the small regular perturbation, and by the Theorem 2, there exists $\varepsilon_1^* > 0, \varepsilon_2^* > 0$, for which the TSPLTISD (1) is stabilizable for any $\varepsilon_1 \in (0, \varepsilon_1^*]$ and $\varepsilon_2 \in (0, \varepsilon_2^*]$ with the composite control (3).

The results obtained are extensively discussed in an illustrative example for the further verification and demonstration of the findings of this study.

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