Then any nontrivial nonnegative solution of (1)–(3) blows up in finite time.

To formulate global existence result for problem (1)–(3) we suppose:

$$f(s) \text{ is a nonnegative locally Hölder continuous function for } s \geqslant 0, \tag{4}$$

$$\text{there exists } p > 0 \text{ such that } f(s) \text{ is a positive nondecreasing function for } s \in (0,p), \tag{5}$$

$$\int_0 \frac{ds}{f(s)} = +\infty, \quad \lim_{s\to 0} \frac{g(s)}{s} = 0, \tag{6}$$

$$\int_0^{+\infty} \big(\alpha(t) + \beta(t)\big)\, dt < +\infty \tag{7}$$

and there exist positive constants γ, t_0 and K such that $\gamma > t_0$ and

$$\int_{t-t_0}^{t} \frac{\beta(\tau)d\tau}{\sqrt{t-\tau}} \leqslant K \quad \text{for } t \geqslant \gamma. \tag{8}$$

Theorem 3. *Let (4)–(8) hold. Then problem (1)–(3) has bounded global solution for small initial datum.*

The results of the talk have been published in [1].

References

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CLASSICAL SOLUTION OF THE INITIAL-VALUE PROBLEM FOR A ONE-DIMENSIONAL QUASILINEAR WAVE EQUATION

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In this report we shall consider the question of global solvability in $[0,\infty)\times\mathbb{R}$ of the initial-value problem

$$\begin{cases} \partial_t^2 u(t,x) - a^2\partial_x^2 u(t,x) + f\big(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)\big) = F(t,x), & (t,x)\in(0,\infty)\times\mathbb{R}, \\ u(0,x) = \varphi(x), \quad \partial_t u(0,x) = \psi(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $a\in(0,\infty)$, φ and ψ are some real-valued functions defined on the real axis.

Theorem 1. *Assume $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$, $F\in C^1([0,\infty)\times\mathbb{R})$, $f\in C^1([0,\infty)\times\mathbb{R}^4)$ and f is Lipschitz continuous in the three last variables. Then there exists a unique classical solution u of the initial-value problem (1).*

Sketch of the proof. We will look for a solution u having the form $u = w + v$ where v solves the homogeneous wave equation

$$\begin{cases} \partial_t^2 v(t,x) - a^2\partial_x^2 v(t,x) = 0, & (t,x)\in(0,\infty)\times\mathbb{R}, \\ v(0,x) = \varphi(x), \quad \partial_t v(0,x) = \psi(x), & x\in\mathbb{R}, \end{cases}$$

and w solves

$$\begin{cases} \partial_t^2 w(t,x) - a^2\partial_x^2 w(t,x) = F(t,x) - f\big(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x)\big), & (t,x)\in(0,\infty)\times\mathbb{R}, \\ w(0,x) = \partial_t w(0,x) = 0, & x\in\mathbb{R}. \end{cases}$$

d'Alembert's Formula [1] lets us write

$$w(t,x) = \frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \Big(F(\tau,\xi) - f\big(\tau,\xi,u(\tau,\xi),\partial_t u(\tau,\xi),\partial_x u(\tau,\xi)\big)\Big)d\xi,\ (t,x)\in[0,\infty)\times\mathbb{R}.$$

Hence our desired solution u must solve the nonlinear integro-differential identity

$$u(t,x) = K[u](t,x),\ \ (t,x)\in[0,\infty)\times\mathbb{R}, \tag{2}$$

where K is the nonlinear mapping defined by the following formula

$$K[u](t,x) = \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi +$$
$$+\frac{1}{2a}\int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \Big(F(\tau,\xi) - f\big(\tau,\xi,u(\tau,\xi),\partial_t u(\tau,\xi),\partial_x u(\tau,\xi)\big)\Big)d\xi,\ (t,x)\in[0,\infty)\times\mathbb{R}.$$

Moreover, it is easy to show that under the conditions of smoothness of the functions φ, ψ, f, F specified in the formulation of this theorem any continuously differentiable solution u of (2) will be a twice continuously differentiable classical solution of the initial-value problem (1).

If $u,\widetilde{u}\in C^1([0,T]\times\mathbb{R})$, $\Omega = \mathrm{Conv}\big\{(0,-m),(0,m),(T,-m+aT),(T,m-aT)\big\}$, $m\in\mathbb{N}$, $T<1$, $m>aT$,

$$\|K[u]-K[\widetilde{u}]\|_{C^1(\Omega)} \leqslant LAT\|u-\widetilde{u}\|_{C^1(\Omega)},$$

where $A=\max\{1,\frac{1}{a}\}$, L is Lipschitz constant of f.

Fix T so small that K is a strict contraction. Observing also that

$$K : C^1([0,T]\times\mathbb{R}) \to C^1([0,T]\times\mathbb{R})$$

if T is small, we see from Banach's Theorem that K has a unique fixed point u which consequently solves the integro-differential identity (2).

We have therefore build a unique classical solution of (1) on $[0,T]\times\mathbb{R}$ provided $T>0$ is sufficiently small. We then extend the solution to the time intervals $[T,2T]$, $[2T,3T]$, etc. using matching conditions, to construct a unique classical solution exiting for all time.

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