GLOBAL CORRECTNESS THEOREM TO GOURSAT PROBLEM FOR INHOMOGENEOUS ADJOINT MODEL TELEGRAPH EQUATION IN THE UPPER HALF-PLANE

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In the upper half-plane $G =]-\infty, +\infty[\times]0, +\infty[$ it is required to find a classical solution and a correctness criterion to the Goursat problem for the adjoint model telegraph equation

$$v_{\tau\tau}(s,\tau) - (a^2(s,\tau)v(s,\tau))_{ss} + (a^{-1}(s,\tau)a_\tau(s,\tau)v(s,\tau))_\tau + (a(s,\tau)a_s(s,\tau)v(s,\tau))_s = f(s,\tau), \quad (1)$$

for all points (s,τ) of curvilinear characteristic triangles $\Delta MPQ \subset G$, and on its curvilinear characteristic sides MP and QM under the Goursat conditions

$$v(s,\tau) = \gamma_1(s,\tau),\ s = h_1\{g_1(x,t),\tau\},\ v(s,\tau) = \gamma_2(s,\tau),\ s = h_2\{g_2(x,t),\tau\}, \quad (2)$$

and at the intersection points $M(x,t)$ of these characteristics, the Goursat data matching condition

$$\gamma_1(x,t) = \gamma_2(x,t),\ \ (x,t) \in G. \quad (3)$$

Here the right-hand side f of the equation and the Goursat data γ_1, γ_2 are given real functions of the variables s and τ, and the coefficient $a(s,\tau) \geqslant a_0 > 0$, $(s,\tau) \in G$, of the equation.

Equation (1) has differential characteristic equations $ds = (-1)^i a(s,\tau)d\tau$, $i = 1, 2$, which correspond in G to two families of characteristics $g_i(s,\tau) = C_i$, $C_i \in \mathbb{R} =]-\infty, +\infty[$, $i = 1, 2$. If the coefficient $a(s,\tau) \geqslant a_0 > 0$, $(s,\tau) \in G$, then the variable τ on the characteristic $g_1(s,\tau) = C_1$, $C_1 \in \mathbb{R}$, strictly decreases, and on characteristic $g_2(s,\tau) = C_2$, $C_2 \in \mathbb{R}$, strictly increases with s. Therefore, the implicit functions $y_i = g_i(s,\tau)$, $s \in \mathbb{R}$, $t \geqslant 0$, have explicit strictly monotonic inverse functions $s = h_i\{y_i,\tau\}$, $\tau \geqslant 0$, and $\tau = h^{(i)}[s,y_i]$, $s \in \mathbb{R}$, $i = 1, 2$. By the definition of inverse functions, they satisfy the inversion identities from [1]:

$$g_i(h_i\{y_i,\tau\},\tau) = y_i\ \forall y_i,\ \ h_i\{g_i(s,\tau),\tau\} = s,\ \ s \in \mathbb{R},\ \ i = 1, 2,$$

$$g_i(s,h^{(i)}[s,y_i]) = y_i\ \forall y_i,\ \ h^{(i)}[s,g_i(s,\tau)] = \tau,\ \ \tau \geqslant 0,\ \ i = 1, 2,$$

$$h_i\{y_i,h^{(i)}[s,y_i]\} = s,\ \ s \in \mathbb{R},\ \ h^{(i)}[h_i\{y_i,\tau\},y_i] = \tau,\ \ \tau \geqslant 0,\ \ i = 1, 2.$$

If the coefficient is $a \in C^2(G)$, then the functions g_i, h_i, $h^{(i)} \in C^2$ by s, τ, y_i, $i = 1, 2$, [1].

The concept of global correctness theorems with explicit solutions and Hadamard correctness criteria (necessary and sufficient conditions) of linear mixed problems was introduced in [2].

Theorem 1. *Let the coefficient be $a(s,\tau) \geqslant a_0 > 0$, $(s,\tau) \in G$, $a \in C^2(G)$. In every triangle $\Delta MPQ \subset G$ the Goursat problem (1)–(3) has a unique and f, γ_1, γ_2-stable classical solution $v \in C^2(\Delta MPQ)$ if and only if $f \in C(G)$ and the smoothness requirements*

$$H_i(s,\tau) \equiv \int_0^\tau f\big(h_i\{g_i(s,\tilde\tau),\tilde\tau\},\tilde\tau\big)\,d\tilde\tau \in C^1(G), \quad i = 1, 2. \quad (4)$$

For $\forall M(x,t) \in G$, this solution to the Goursat problem (1)–(3) in $\Delta MPQ \subset G$ is the function

$$v(s,\tau;x,t)=\big\{[a(\tilde s,\tilde\tau)\gamma_2(\tilde s,\ \tilde\tau)-F(\tilde s,\tilde\tau)]|_{\tilde s=h_2\{g_2(x,t),\,\tilde\tau\}}|_{\tilde\tau=\tau_1(g_1(s,\tau))}+$$
$$+[a(\tilde s,\tilde\tau)\gamma_1(\tilde s,\tilde\tau)-F(\tilde s,\tilde\tau)]|_{\tilde s=h_1\{g_1(x,t),\,\tilde\tau\}}|_{\tilde\tau=\tau_2(g_2(s,\tau))}- \tag{5}$$
$$-[a(\tilde s,\tilde\tau)\gamma_2(\tilde s,\tilde\tau)-F(\tilde s,\tilde\tau)]|_{\tilde s=h_2\{g_2(x,t),\,\tilde\tau\}}|_{\tilde\tau=\tau_1(g_1(x,t))}+F(s,\tau)\big\}/a(s,\tau),\ (s,\tau)\in\Delta MPQ\subset G,$$

where the particular classical solution of the equation (1) is the product of $1/a(s,\tau)$ *by*

$$F(s,\tau)=\frac{1}{2}\int\limits_0^{\tau}d\tilde\tau\int\limits_{h_2\{g_2(s,\tau),\tilde\tau\}}^{h_1\{g_1(s,\tau),\tilde\tau\}}f(\tilde s,\tilde\tau)\,d\tilde s,$$

$\tau_1(y)$, $\tau_2(z)$ *are inverse functions to functions*

$$y=g_1\big(h_2\{g_2(x,t),\tau\},\tau\big),\quad z=g_2\big(h_1\{g_1(x,t),\tau\},\tau\big).$$

Sketch of the proof. The Goursat problem (1)–(3) is solved by the characteristic method. To find the general integral of the equation (1) on G, in it we pass to new variables

$$\xi=g_1(s,\tau),\quad \eta=g_2(s,\tau) \tag{6}$$

with non-degenerate Jacobian $J(s,\tau)=\xi_s\eta_\tau-\xi_\tau\eta_s\neq 0$ in G, since $a(s,\tau)\geqslant a_0>0$, $(s,\tau)\in G$. By replacing (6) the wave equation (1) for the new function $\tilde v(\xi,\eta)=v(s(\xi,\eta),\tau(\xi,\eta))$ reduced to canonical form

$$\big(\tilde a(\xi,\eta)\tilde v(\xi,\eta)\big)_{\xi\eta}=\tilde f(\xi,\eta)/[2J(s,\tau)],\ (\xi,\eta)\in\widetilde G, \tag{7}$$

where the coefficient $\tilde a(\xi,\eta)=a(s(\xi,\eta),\tau(\xi,\eta))$, the right-hand side $\tilde f(\xi,\eta)=f(s(\xi,\eta),\tau(\xi,\eta))$ and the set $\widetilde G=\{(\nu,\rho):h_2\{\rho,0\}\leqslant h_1\{\nu,0\},\ \nu,\rho\in\mathbb{R}\}$.

Integrating equation (7) over the triangle $\triangle\tilde M\tilde P\tilde Q\subset\widetilde G$, we find its general integral

$$\tilde a(\xi,\eta)\tilde v(\xi,\eta)=\tilde f_1(\xi)+\tilde f_1(\eta)+\tilde F(\xi,\eta), \tag{8}$$

where triangle $\triangle\tilde M\tilde P\tilde Q$ is the image of triangle $\triangle MPQ$ under mapping (6), the function $\tilde F$ is

$$\tilde F(\xi,\eta)=\frac{1}{2}\int\limits_{g_2(h_1\{\xi,0\},0)}^{\eta}d\rho\int\limits_{g_1(h_2\{\rho,0\},0)}^{\xi}\tilde f(\nu,\rho)\,J(\nu,\rho)\,d\nu,$$

$J(\xi,\eta)=s_\xi\tau_\eta-s_\eta\tau_\xi\neq 0$ is the Jacobian of the inversely replacing to variables (6) on the set $\widetilde G$ and the product of these Jacobians is $J(\xi,\eta)J(s,\tau)=1$. From the general integral (8) by inverse replacement to variables (6) we derive the general integral of the equation (1)

$$v(s,\tau)=\big[\tilde f_1\big(g_1(s,\tau)\big)+\tilde f_1\big(g_2(s,\tau)\big)+F(s,\tau)\big]/a(s,\tau), \tag{9}$$

where $\tilde f_1$ and $\tilde f_2$ are any twice continuously differentiable functions of ξ and η of the form

$$\tilde f_1(\xi)=f_1(\xi)+f_2\big(g_2(x,t)\big),\quad \tilde f_2(\eta)=f_2(\eta)-f_2\big(g_2(x,t)\big).$$

We substitute the general integral (9) into the Goursat conditions (2) and obtain the formal solution (5). Then we show the necessity and sufficiency of the smoothness $f\in C(G)$ and (4).

Corollary 1. *If the right-hand side f of the equation (1) does not depend on s or τ in G, then the assertion of Theorem 1 is valid without integral smoothness requirements (4).*

Here, for a continuous right-hand side f in τ or s smoothness (4) always holds.

Remark 1. In the case $F = 0$, solution (5) serves as the Riemann function into Riemann formula of solutions to all mixed problems for the inhomogeneous model telegraph equation [3].

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