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ON SPECTRAL PROBLEM FOR MHD EQUATION

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We are interesting the equations that govern for the simplest cases the liquid metal flow magnetohydrodynamics that, in particular, can be used in the nuclear fusion reactor.

We consider the case of a spacially periodic, incompressible, conducting fluid in a $3\,D$ cubic box Ω of size $L = L_{\text{box}}$ under imposed homogeneous and steady magnetic field B_0 aligned with the vertical direction e_z. The governing equations can be reduced to a single one involving the velocity and pressure only (see [1,2]) as follows

$$\frac{\partial}{\partial t}u(\mathbf{x},t) + (u\cdot\nabla)u + \nabla p = \nabla^2 u - Ha^2\nabla^{-2}\frac{\partial^2 u}{\partial z^2} + Grf(x,t), \quad \nabla\cdot u = 0,$$

where following notations are used $u(x,t)$ is the velocity-vector of the flow, $f(x,t)$ is the external forcing, $x=(x,y,x)$ is the spatial variable, t is time, $Ha = L_{\text{ref}}B_0\sqrt{\dfrac{\sigma}{\rho\nu}}$ is the Hartmann number and $Gr = \dfrac{L_{\text{ref}}^{3-d/2}}{\nu^2}||f||$ is the Grashoff number (d is number of spatial dimensions), ρ is the density, p is the pressure, ν is the viscosity, σ is the electrical conductivity, B_0 is the imposed magnetic field, $Re = \frac{UL_{\text{int}}}{\nu}$ is Reynolds number. The addition of periodic boundary conditions and zero initial condition $u(x,0)=0$ completely determine the problem.

In order to rewrite the considered problem in the abstract form as an initial boundary value problem and identify the operator relevant to the problem, we first need to identify the functional spaces where the solution of our problem are to be sought.

Following [3], let $H^m(\Omega)$ be the Sobolev space of functions from $L_2(\Omega)$ whose derivatives of order up to m belong to $L_2(\Omega)$. $\big(H^m(\Omega)\big)^3$ is the space of three-dimensional vector fields with components from $H^m(\Omega)$. By $L_p(\Omega;\mathcal{B})$, $1\leqslant p<\infty$, we denote the set of functions v defined on the domain $\Omega\subset\mathbb{R}^n$ with images in the given Banach space $\mathcal{B}$, for which the norm $||v||_{L_p(\Omega;\mathcal{B})} = \left(\int\limits_\Omega ||v||^p_{\mathcal{B}}dE\right)^{1/p}$ is finite. Then, define the needed spaces V, V^0, V^1 and V^2 as follows:

$$V = \left\{v(x)\in\big(H^1(\Omega)\big)^3 : \operatorname{div} v = 0 \text{ in } \Omega,\ v \text{ satisfies the boundary conditions on } \partial\Omega\right\},$$

$$V^0 \text{ is the closure of } V \text{ in } \big(L_2(\Omega)\big)^3, \quad V^1 \text{ is the closure of } V \text{ in } \big(H^1(\Omega)\big)^3,$$

$$V^2 = V^1\cap\big(H^2(\Omega)\big)^3.$$

Then, assuming $f(\cdot) \in L_2(0,T;\, V^0)$ and $u_0 \in V^1$ and using the Helmholtz-Leray decomposition of vector field u (see [3]), we obtain the following Cauchy problem for the considered function u:

$$\dot{u}(t) = Lu + B(u) + f(t), \quad u(t)\big|_{t=0} = u_0,$$

where $u(t) = u(\cdot,t)$ and $u(\cdot) \in \{L_2(0,T;\, V^2) : \dot{u}(t) \in L_2(0,T;\, V^0)\}$, $f(t) = f(\cdot,t)$ and the operators L and B are defined as:

$$Lu = \frac{1}{Re}\mathcal{P}\big(\nabla^2 u - Ha^2 \nabla^{-2}\frac{\partial^2 u}{\partial z^2}\big) \quad \forall u \in D(L),$$

$$B(u) = \mathcal{P}(u \cdot \nabla)u \quad \forall u \in D(L),$$

where $D(L) = D(B) = V \cap \big(H^2(\Omega)\big)^3$ and $\mathcal{P}$ denotes the Leray projector $\mathcal{P}\colon (L_2(\Omega))^3 \to V^0$.

The principle of spectral methods consists in looking for a solution for the physical variables (here the space x and time t dependent velocity field $u(x,t)$ and pressure $p(x,t)$) under the form of a finite expansion over a basis of functions (e_n), $1 \leqslant n \leqslant N$, that spans the space of solutions of the PDE. The novelty is that instead of using bases like the usual Chebyshev polynomials, that are easy to implement but incur heavy computational costs in order to resolve the problem, we use a basis obtained from the eigenvalue problem of the linear part of the governing equations. This method used in [4] and shown their efficiency and minimal computational costs for numerical solutions in realistic experiments.

Finally, the problem associated with the liquid metal flow magnetohydrodynamics is to study in the relevant function spaces the dissipation operator of the form

$$D_{Ha} = \left(\triangle - Ha^2 \triangle^{-1}\frac{\partial^2}{\partial z^2}\right), \quad \triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

We shall first solve the eigenvalue problem for this operator

$$D_{Ha} f = \lambda f, \quad \left(\triangle - Ha^2 \triangle^{-1}\frac{\partial^2}{\partial z^2}\right) f = \lambda f,$$

that is equivalent to the eigenvalue problem

$$\left(\triangle^2 - Ha^2 \frac{\partial^2}{\partial z^2}\right) f = \lambda \triangle f$$

with periodic conditions in x, y directions and walls in $z = \pm 1$, and where

$$\triangle^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + 2\frac{\partial^4}{\partial x^2 \partial z^2} + 2\frac{\partial^4}{\partial y^2 \partial z^2}.$$

The last equation for some cases can be rewritten as follows

$$Z^{(4)}_{zzzz} - 2\left[k_\perp^2 + \frac{1}{2}(Ha^2 + \lambda)\right] Z^{(2)}_{zz} + \left[k_\perp^4 + \lambda k_\perp^2\right] Z = 0.$$

where $k_\perp = \sqrt{k_x^2 + k_y^2}$. Hence, the characteristic equation for the corresponding eigenvalues is

$$\mu^4 - 2\left[k_\perp^2 + \frac{1}{2}(Ha^2 + \lambda)\right]\mu^2 + \left[k_\perp^4 + \lambda k_\perp^2\right] = 0.$$

The proper study of these roots gives an ability to calculate the needed eigenfunctions. This problem is investigated in the given paper.

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MULTI-DIMENSIONAL GENERAL INTEGRAL TRANSFORMATION WITH SPECIAL FUNCTIONS IN THE WEIGHTED SPACE OF SUMMABLE FUNCTIONS

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Multidimensional integral transform

$$\big(\mathrm{K}f\big)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int\limits_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{x}\mathbf{t}] f(\mathbf{t})\, \mathrm{d}\mathbf{t} \quad (\mathbf{x} > 0) \tag{1}$$

is studied. Here (see, for example, [1] Section 28.4; [2], ch. 1; [3], [4])

$$\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n; \quad \mathbf{t} = (t_1, t_2, ..., t_n) \in \mathbb{R}^n,$$

$\mathbb{R}^n$ be the n-dimensional Euclidean space; $\mathbf{x}\cdot\mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x}\cdot\mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for **1**= (1,1,...,1). The expression $\mathbf{x} > \mathbf{t}$ means that

$$x_1 > t_1, \quad x_2 > t_2, \quad ..., \quad x_n > t_n,$$

the nonstrict inequality $\geqslant$ has similar meaning;

$$\int\limits_{\mathbf{0}}^{\infty} = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \cdots \int\limits_{0}^{\infty};$$

by $\mathbb{N} = \{1, 2, ...\}$ we denote the set of positive integers,

$$\mathrm{N}_0 = \mathbb{N} \cup \{0\}, \quad \mathrm{N}_0^n = \mathrm{N}_0 \times \mathrm{N}_0 \times ... \times \mathrm{N}_0.$$

$$\mathbf{k} = (k_1, k_2, ..., k_n) \in \mathrm{N}_0^n = \mathrm{N}_0 \times ... \times \mathrm{N}_0 \ \ (k_i \in \mathrm{N}_0, \ \ i = 1, 2, ..., n)$$

is a multi-index with $\mathbf{k}! = k_1! \cdots k_n!$ and $|\mathbf{k}| = k_1 + k_2 + ... + k_n$.

$$\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n, \ \mathbf{x} > 0\};$$

for $l = (l_1, l_2, ..., l_n) \in \mathbb{R}^n_+$

$$\mathbf{D}^l = \frac{\partial^{|l|}}{(\partial x_1)^{l_1} \cdots (\partial x_n)^{l_n}}; \qquad \mathbf{dt} = dt_1 \cdot dt_2 \cdots dt_n;$$