MULTI-DIMENSIONAL MODIFIED G-TRANSFORMATIONS AND THEIR SPECIAL CASES

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Three multidimensional integral transforms

$$\left(\mathrm{G}^{1}_{\sigma,\kappa}f\right)(\mathbf{x}) = \mathbf{x}^{\sigma}\int\limits_{0}^{\mathbf{x}} \mathrm{G}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\frac{\mathbf{x}}{\mathbf{t}}\left|\begin{matrix}(\mathbf{a}_i)_{1,p}\\(\mathbf{b}_j)_{1,q}\end{matrix}\right.\right]\mathbf{t}^{\kappa}f(\mathbf{t})\,\frac{\mathbf{dt}}{\mathbf{t}} \quad (\mathbf{x}>0); \tag{1}$$

$$\left(\mathrm{G}^{1}_{\sigma,\kappa;\,\delta}f\right)(\mathbf{x}) = \mathbf{x}^{\sigma}\int\limits_{0}^{\mathbf{x}} \mathrm{G}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\frac{\mathbf{x}^{\delta}}{\mathbf{t}^{\delta}}\left|\begin{matrix}(\mathbf{a}_i)_{1,p}\\(\mathbf{b}_j)_{1,q}\end{matrix}\right.\right]\mathbf{t}^{\kappa}f(\mathbf{t})\,\frac{\mathbf{dt}}{\mathbf{t}} \quad (\mathbf{x}>0); \tag{2}$$

$$\left({}_1\mathrm{I}_{\sigma,\,\omega;\,\zeta}f\right)(\mathbf{x}) = \mathbf{x}^{\sigma}\int\limits_{0}^{\mathbf{x}} \frac{\left(\mathbf{x}^{\zeta}-\mathbf{t}^{\zeta}\right)^{c-1}}{\Gamma(c)}\,{}_2\mathrm{F}_1\left(a,b;c;1-\frac{\mathbf{x}^{\zeta}}{\mathbf{t}^{\zeta}}\right)\mathbf{t}^{\omega}f(\mathbf{t})\,d\mathbf{t} \quad (\mathbf{x}>\mathbf{0}) \tag{3}$$

are studied. Here (see, for example, [1], [2] Section 28.4; [3], ch. 1; [4]; [5])

$$\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n,\quad \mathbf{t}=(t_1,t_2,\ldots,t_n)\in\mathbb{R}^n,$$

$$\mathbb{R}^n \text{ – Euclidean } n\text{-space},\quad \mathbb{N}=\{1,2,...\},\quad \mathrm{N}_0=\mathbb{N}\cup\{0\},\quad \mathrm{N}_0^n=\mathrm{N}_0\times\mathrm{N}_0\times...\times\mathrm{N}_0;$$

$$\mathbf{m}=(m_1,m_2,...,m_n)\in\mathrm{N}_0^n \quad\text{and}\quad m_1=m_2=...=m_n;$$

$$\mathbf{n}=(\overline{n}_1,\overline{n}_2,...,\overline{n}_n)\in\mathrm{N}_0^n \quad\text{and}\quad \overline{n}_1=\overline{n}_2=...=\overline{n}_n;$$

$$\mathbf{p}=(p_1,p_2,...,p_n)\in\mathrm{N}_0 \quad\text{and}\quad p_1=p_2=...=p_n;$$

$$\mathbf{q}=(q_1,q_2,...,q_n)\in\mathrm{N}_0^n \quad\text{and}\quad q_1=q_2=...=q_n \quad (0\leqslant\mathbf{m}\leqslant\mathbf{q},\quad 0\leqslant\mathbf{n}\leqslant\mathbf{p});$$

$$\sigma=(\sigma_1,\sigma_2,...,\sigma_n)\in\mathbb{C}^n;\quad \kappa=(\kappa_1,\kappa_2,...,\kappa_n)\in\mathbb{C}^n;$$

the function

$$\mathrm{G}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\mathbf{z}\left|\begin{matrix}(\mathbf{a}_i)_{1,p}\\(\mathbf{b}_j)_{1,q}\end{matrix}\right.\right] = \prod_{k=1}^{n}\mathrm{G}^{m_k,\overline{n}_k}_{p_k,q_k}\left[z_k\left|\begin{matrix}(a_{i_k})_{1,p_k}\\(b_{j_k})_{1,q_k}\end{matrix}\right.\right]$$

is the product of the G-functions $\mathrm{G}^{m_k,\overline{n}_k}_{p_k,q_k}[z_k]$ $(k=1,2,...n)$ [6];

$$a=(a_1,a_2,\ldots,a_n),\ b=(b_1,b_2,\ldots,b_n),\ c=(c_1,c_2,\ldots,c_n)\in\mathbb{R}^n,\ 0<c_j<1,\ j=\overline{1,n};$$

$$\sigma=(\sigma_1,\sigma_2,...,\sigma_n)\in\mathbb{R}^n;\quad \omega=(\omega_1,\omega_2,...,\omega_n)\in\mathbb{R}^n;\quad \zeta=(\zeta_1,\zeta_2,...,\zeta_n)\in\mathbb{R}^n_+;$$

$$(\mathbf{x}^{\zeta}-\mathbf{t}^{\zeta})^{c-1}=\prod_{j=1}^{n}(x_j^{\zeta_1}-t_j^{\zeta_1})^{c_j-1};\qquad \int\limits_0^{\mathbf{x}}=\int\limits_0^{x_1}\int\limits_0^{x_2}\cdots\int\limits_0^{x_n};$$

the expression $\mathbf{x}\geqslant\mathbf{t}$ means $x_1\geqslant t_1,\ x_2\geqslant t_2,\ \ldots,\ x_n\geqslant t_n$; $d\mathbf{t}=dt_1\cdot dt_2\cdots dt_n$; $F\,(a,b;c;\mathbf{z})$ is a function of the form [7]:

$$F\,(a,b;c;\mathbf{z})=\prod_{j=1}^{n}{}_2F_1\,(a_j,b_j;c_j;z_j),$$

${}_2F_1\,(a_j,b_j;c_j;z_j)$ $(j=1,2,...,n)$ are the Gauss hypergeometric functions.

Our report is devoted to the study of tranforms (1)–(3) in the weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{r}}$ summabe functions $f(\mathbf{x})=f(x_1,x_2,\ldots,x_n)$ on $\mathbb{R}^{\mathbf{n}}_{+}=\{\mathbf{x}:\mathbf{x}\in\mathbb{R}^n \mid x_1>0,x_2>0,\ldots,x_n>0\}$, such that

$$\|f\|_{\overline{\nu},\overline{r}}=\Bigg\{\int\limits_{\mathbb{R}^1_+} x_n^{\nu_n\cdot r_n-1}\Big\{\cdots\Big\{\int\limits_{\mathbb{R}^1_+} x_2^{\nu_2\cdot r_2-1}\times$$

$$\times\Big[\int\limits_{\mathbb{R}^1_+} x_1^{\nu_1\cdot r_1-1}|f(x_1,...,x_n)|^{r_1}dx_1\Big]^{r_2/r_1}dx_2\Big\}^{r_3/r_2}\cdots\Big\}^{r_n/r_{n-1}}dx_n\Bigg\}^{1/r_n}<\infty \tag{4}$$

$\big(\overline{r}=(r_1,r_2,...,r_n)\in\mathbb{R}^n,\ 1\leqslant\overline{r}<\infty,\ \overline{\nu}=(\nu_1,\nu_2,\ldots,\nu_n)\in\mathbb{R}^n,\ \nu_1=\nu_2=\ldots=\nu_n\big)$.

The functional properties of the integral transformations (1)–(3) in spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ $\big(\overline{2}=(2,2,\ldots,2),\ \overline{\nu}=(\nu_1,\nu_2,\ldots,\nu_n)\in\mathbb{R}^n,\ \nu_1=\nu_2=\ldots=\nu_n\big)$ were studied in [8]. We continue this research. Constructed the $\mathfrak{L}_{\overline{\nu},\overline{r}}$-theory of three multidimensional integral transformations (1)–(3) with special functions in the kernels: the G-function and the Gauss hypergeometric function. Conditions are obtained for the q and one-to-oneness of the operators of such transformations from one spaces (4) of integrable functions $\mathfrak{L}_{\overline{\nu},\overline{r}}$ to others, and analogs of the formula for integration by parts are proved. For the transformations under consideration, various integral representations are established and inversion formulas are derived.

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