UDC 519.2

INDEPENDENT RE-TESTING AS THE ASYMPTOTIC GAUSSIAN RANDOM PROCESS

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Independent re-tests are considered as a random process with discrete time. The asymptotic behavior of the process through the method of moments is set and the contribution asymmetries and excesses of arbitrary orders in the deviation of Bernoulli distribution from the asymptotic formula, appearing in the local theorem of Laplace is defined. It is shown that the latter cannot be applied to calculate the number success probabilities which are far from their most probable values.

The probability m of progress in n independent tests is defined by Bernoulli's formula [1]

$$P(n,m) = C_n^m p^m (1-p)^{n-m}\,, \tag{1}$$

in which C_n^m - numbers of combinations, p - probability of success in one experience. For larger n calculation C_n^m becomes a problem. In particular MathCAD cannot work with the numbers exceeding 10^{307}. To bypass this difficulty the limiting theorems of Laplace are used which, however, not always provide the required accuracy at all and are not applicable for m far from the most probable values.

At the same time, if n grows, Bernoulli's scheme can be considered as a casual process with discrete time. At the same time it is m convenient to investigate evolution of the distribution law by method of the moments, realized in [2-5] for the description of dynamics of a sorption and diffusion. Thus, it is possible not only to find out a process asymptotic, but also to define the amendments to it caused by asymmetries and excesses of the distribution law. This publication is devoted to the solution of these tasks.

It is known that all information on the distribution law m contains in the initial moments

$$\nu_k(n) = \sum_{m=0}^{n} m^k \cdot P(n,m)\,. \qquad (k = 0, 1, ...) \tag{2}$$

In particular,

$$\nu_1(n) = np\,, \qquad \sigma(n)^2 = \nu_2(n) - \nu_1(n)^2 = np(1-p) \tag{3}$$

- respectively the expectation and dispersion m.

To define an asymptotic of process it is necessary to investigate asymmetries and excesses m:

$$A_{2k-1}(n) = \frac{\mu_{2k-1}(n)}{\sigma(n)^{2k-1}}\,, \qquad E_{2k}(n) = \frac{\mu_{2k}(n)}{\sigma(n)^{2k}} - \omega(k)\,, \qquad (k = 2, 3, ...) \tag{4}$$

where

$$\omega(k) = \int_{-\infty}^{\infty} x^{2k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 \cdot 3 \cdot ... \cdot (2k-1)\,, \tag{5}$$

$$\mu_k(n) = \sum_{m=0}^{n} (m - \nu_1(n))^k \cdot P(n,m) \tag{6}$$

- central momen. It is convenient to find them by means of characteristic function:

$$\theta(\tau,n) = \sum_{m=0}^{n} e^{i\tau(m-np)} \cdot P(n,m) = \left(p\,e^{i\tau q} + q e^{-i\tau p}\right)^n\,, \tag{7}$$

$$\mu_k(n) = i^{-k} \theta^{(k)}(0,n)\,, \tag{8}$$

where i - imaginary unit, (k) - derivative order on τ.

From (4) – (8) follows, that

$$A_{2k-1}(n,p)=\sum_{l=1}^{k-1}\frac{f_{2l-1,2k-1}(p)}{\sigma(n)^{2l-1}}, \qquad E_{2k}(n,p)=\sum_{l=1}^{k-1}\frac{f_{2l,2k}(p)}{\sigma(n)^{2l}}, \tag{9}$$

where

$$f_{1,3}(p)=1-2p, \qquad f_{2,4}(p)=6p^2-6p+1, \qquad f_{1,5}(p)=10(1-2p), \dots \tag{10}$$

- independent from n decomposition values. It agrees (3), (9) at $n\to\infty$ asymmetry and an excess disappear that is characteristic of the asymptotic Gaussian process

$$P(n,m)\xrightarrow[n\to\infty]{}P_0(n,m)=\frac{1}{\sqrt{2\pi}\sigma(n)}e^{-\frac{x(n,m)^2}{2}}, \qquad x(n,m)=\frac{m-np}{\sigma(n)}. \tag{11}$$

Let's find out a contribution in

$$\Delta(n,m)=P(n,m)-P_0(n,m) \tag{12}$$

asymmetries and excesses of the random orders. Taking into account (9) we will write down

$$\Delta(n,m)=P_0(n,m)\sum_{l=1}^{\infty}\frac{\varphi_l(x(n,m))}{\sigma(n)^l}=\Delta_1(n,m)+\Delta_2(n,m)+\dots, \tag{13}$$

where senior degree of polynoms

$$\varphi_{2l-1}(x)=\sum_{i=0}^{l}x^{2i+1}C_{2l-1,\,2i+1}, \qquad \varphi_{2l}(x)=\sum_{i=0}^{l+1}x^{2i}C_{2l,\,2i} \qquad (l=1,2,\dots) \tag{14}$$

is equal to quantity linearly of the independent equations received by means of following from (6), (11) – (13) identities

$$\frac{\mu_k(n)}{\sigma(n)^k}=\sum_{m=0}^{n}x(n,m)^k\cdot P_0(n,m)\left(1+\sum_{l=1}^{\infty}\frac{\varphi_l(x(n,m))}{\sigma(n)^l}\right), \qquad (k=0,1,\dots) \tag{15}$$

for calculation appearing in (14) coefficients $C_{l,i}$. At $n\to\infty$ the sum on m in (15) becomes integral

$$\Delta x(n,m)=x(n,m+1)-x(n,m)=\sigma(n)^{-1}\xrightarrow[n\to\infty]{}0,$$

$$x(n,0)=-\sqrt{np/q}\xrightarrow[n\to\infty]{}-\infty \qquad x(n,n)=\sqrt{nq/p}\xrightarrow[n\to\infty]{}\infty,$$

therefore

$$\frac{\mu_k(n)}{\sigma(n)^k}\xrightarrow[n\to\infty]{}\int_{-\infty}^{\infty}x^k\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left(1+\sum_{l=1}^{\infty}\frac{\varphi_l(x)}{\sigma(n)^l}\right)dx. \tag{16}$$

The equations relatively $C_{l,i}$ turn out equating in (16) expressions at identical degrees $\sigma(n)$ .As the integral from an odd function is equal in the symmetric limits to zero, even and odd k will be considered separately

$$\frac{\mu_{2k+1}(n)}{\sigma(n)^{2k+1}}\xrightarrow[n\to\infty]{}\int_{-\infty}^{\infty}x^{2k+1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\sum_{l=1}^{\infty}\frac{\varphi_{2l-1}(x)}{\sigma(n)^{2l-1}}dx, \tag{17}$$

$$\frac{\mu_{2k}(n)}{\sigma(n)^{2k}}\xrightarrow[n\to\infty]{}\int_{-\infty}^{\infty}x^{2k}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left(1+\sum_{l=1}^{\infty}\frac{\varphi_{2l}(x)}{\sigma(n)^{2l}}\right)dx. \tag{18}$$

From (17) taking into account (5), (14) follows

$$\frac{\mu_{2k+1}(n)}{\sigma(n)^{2k+1}}\xrightarrow[n\to\infty]{}\sum_{l=1}^{\infty}\frac{1}{\sigma^{2l-1}}\sum_{i=0}^{l}C_{2l-1,2i+1}\omega(k+i+1), \tag{19}$$

Then

$$C_{2l-1,2i+1} = \frac{f_{2l-1,2l+1}(p)}{\rho(l)} \cdot \frac{(-1)^{i+l} C_l^i}{\omega(i+1)}, \qquad \begin{pmatrix} l = 1, 2, ... \\ i = 0, 1, ..., l. \end{pmatrix} \tag{20}$$

where

$$\rho(l) = 2 \cdot 4 \cdot ... \cdot 2l \tag{21}$$

- works of even numbers, C_l^i - numbers of combinations.

Let's similarly consider in (16) even k. From (18) taking into account (5), (14) follows

$$\frac{\mu_{2k}(n)}{\sigma(n)^{2k}} - \omega(k) \xrightarrow[n\to\infty]{} \sum_{l=1}^{\infty} \frac{1}{\sigma^{2l}} \sum_{i=0}^{l+1} C_{2l,2i}\,\omega(k+i). \tag{22}$$

Then for even polynoms

$$C_{2l,2i} = \frac{f_{2l,2(l+1)}(p)}{\rho(l+1)} \cdot \frac{(-1)^{i+l+1} C_{l+1}^i}{\omega(i)}. \qquad \begin{pmatrix} l = 1, 2, ... \\ i = 0, 1, ..., l+1. \end{pmatrix} \tag{23}$$

Thus, formulas (3), (11) – (14), (20), (21), (23) allow to replace expression, which is hard to calculate at larger n, for probability of achievements in Bernoulli's scheme. Results of the calculations executed by them in a graphic form are presented in fig. 1. It is visible that even for $n = 5$ the error of a zero approximation (fig. 1a) almost completely disappears (fig. 1b) at the accounting of asymmetry and an excess of minimum orders ($P_2(n,m) = P_0(n,m) + \Delta_1(n,m) + \Delta_2(n,m)$ see (12), (13)).

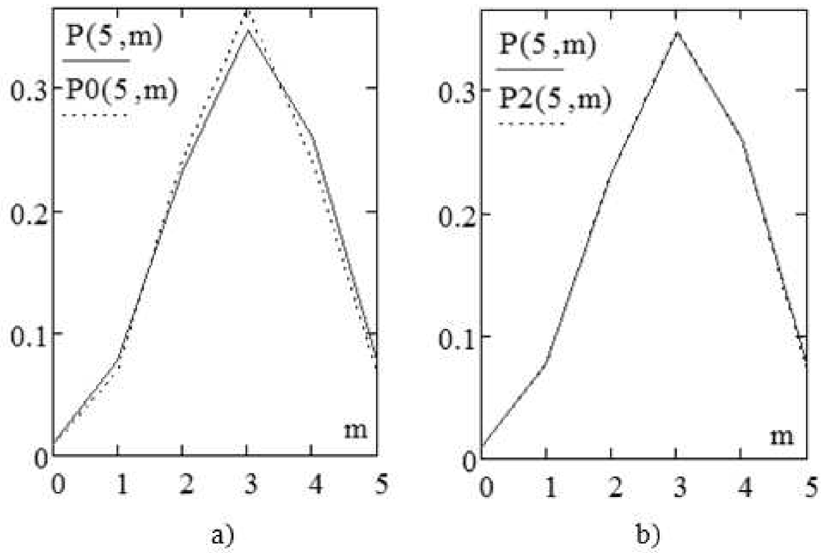

Fig. 1. Influence of asymmetry and excess of minimum orders on probability of m progress in 5 experiences of Bernoulli"s scheme with $p = 0.6$

According to (13), (14) it is possible to use the local theorem of Laplace (answering to a zero approximation in decomposition (12), (13)),if $|x(n,m)|^{n+2} = o\left(\sigma(n)^n\right)$. In particular, for m progress, far from the most probable number, $|x(n,m)|$ at increase in number of experiences grows in proportion $\sqrt{n}$ (see (3), (11)) that leads to a divergence of decomposition (13) on the inverse degrees $\sigma(n)$. As a result, the relative accuracy of the asymptotic formula (11) appearing in the local theorem of Laplace beyond all bounds increases (fig. 2) and it is impossible to use it that, however, not always makes a reservation [1].

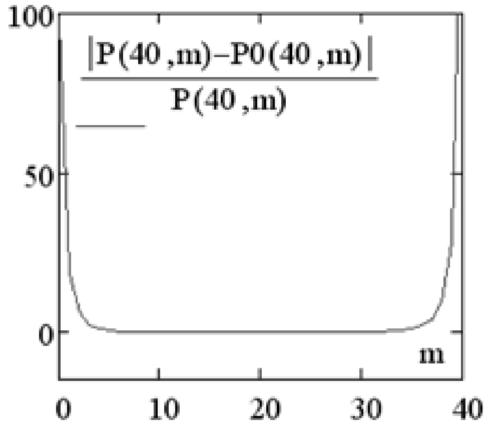

Fig. 2. The relative accuracy of a zero approximation for $p = 0.6$

Thus, it is offered to consider independent retests as casual process with discrete time established. The asymptotic of process through the method of the moments was established and the contribution of asymmetries and excesses of the random orders to a deviation of distribution of Bernoulli from the asymptotic formula appearing in the local theorem of Laplace was defined. It is shown that the latter cannot be applied for the calculation of the number success probabilities which are far from their most probable values.

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