UDC 517.983

SOLUTION OF ONE CLASS OF ONE-DIMENSIONAL INTEGRAL EQUATION WITH LEGENDRE FUNCTION OF THE FIRST KIND IN THE KERNEL

DANIL KOROTKII, OKSANA SKOROMNIK
Polotsk State University, Belarus

This paper is devoted to the study of one class of one-dimensional integral equation

$$\int_a^x \left(x^2-t^2\right)^{-\mu/2} \mathrm{P}_\nu^\mu\left(\frac{x}{t}\right) f(t)dt = g(x) \ \ (x>a),$$

involving the Legendre function of the first kind $\mathrm{P}_\nu^\mu(z)$ *with real* $\mu, \nu, 0<\mu<1$, *in the kernel. By Tamarkin's method the solution of the investigating equation in the closed form are obtained, and necessary and sufficient conditions for its solvability in the space of summable functions on a finite interval* $[a,b]$ *of the real line are given.*

Introduction.

We consider the integral equation

$$\int_a^x \left(x^2-t^2\right)^{-\mu/2} P_\nu^\mu\left(\frac{x}{t}\right) f(t)dt = g(x) \ \ (x>a), \tag{1}$$

containing the Legendre function of the first kind $\mathrm{P}_\nu^\mu\left(\frac{x}{t}\right)$ in the kernel, see [1, formulas 3.2(3) and 3.4(6)] and [2, Section 2.1]; $\mu\in R, \nu\in R, 0<\mu<1$. By Tamarkin's method the solution of the equation (1) in the closed form are obtained, and necessary and sufficient conditions for its solvability in the space of summable functions on a finite interval $[a,b]$ of the real line are given.

We need the generalization [1, 2.4(3)]:

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)} \int_0^1 \frac{x^{s-1}(1-x)^{c-s-1}}{(1-xz)^{a'}} \, {}_2F_1(a-a',b;s;xz) \, {}_2F_1\left(a',b-s;c-s;\frac{(1-x)z}{1-xz}\right)dx, \tag{2}$$

where ${}_2F_1(a,b,c,z)$ is the Gauss hypergeometric function defined for complex $a,b,c\in C$ and $|z|<1$ by the hypergeometric series ${}_2F_1(a,b,c,z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$ with the corresponding analytic continuation

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,$$

for $z\in C, 0<\operatorname{Re}b<\operatorname{Re}c, (|\arg(1-z)|<\pi, z\neq 1)$, see [1, 2.1(2) and 2.1.(10)]), here $(z)_n$ is the Pochhammer symbol, that is, $(z)_0\equiv 1$, $(z)_n = z(z+1)...(z+n-1)$ $(z\in C; n\in N)$;

the representation [1, 3.2(9.24)]:

$$\mathrm{P}_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} 2^\mu (z^2-1)^{-\frac{\mu}{2}} z^{\nu+\mu} F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}; 1-\mu; 1-\frac{1}{z^2}\right), \operatorname{Re}(z)>0, \left|1-\frac{1}{z^2}\right|<1; \tag{3}$$

and the following formula [2, formula (1.32)]

$$\int_a^b dx \int_a^x f(x,y)dy = \int_a^b dy \int_y^b f(x,y)dx, \tag{4}$$

provided that one of the multiple integrals in (4) converges absolutely.

Solution in the closed–form.

First, we give a formal solution of the equation (1). Using the representation (3) for $P_\nu^\mu\left(\frac{x}{t}\right)$, we rewrite equation (1) in the form

$$\frac{2^\mu x^{\nu+\mu}}{\Gamma(1-\mu)}\int_0^x (x^2-t^2)^{(1-\mu)-1} t^{-\nu} F\left(-\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2};1-\mu;\frac{(x^2-t^2)}{x^2}\right) f(t)dt = g(x).$$

Replacing x by t and t by u, multiplying both sides of the resulting equality by

$$t(x^2-t^2)^{\mu-1} F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{(x^2-t^2)}{x^2}\right),$$

integrating, and changing the order of integration according to the formula (4), we obtain:

$$\frac{2^{\mu-1}}{\Gamma(1-\mu)}\int_a^x u^{-\nu} f(u)du \int_u^x t^{\nu+\mu}(x^2-t^2)^{\mu-1}(t^2-u^2)^{(1-\mu)-1}$$

$$\times F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{(x^2-t^2)}{x^2}\right) F\left(-\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2};1-\mu;\frac{t^2-u^2}{t^2}\right) 2tdt = \tag{5}$$

$$= \int_a^x (x^2-t^2)^{\mu-1} F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{(x^2-t^2)}{t^2}\right) t\, g(t)dt.$$

To calculate the inner integral in (5), we introduce the new variables $s=\frac{x^2-t^2}{x^2-u^2}$. Using the formula (2), we see that the inner integral in (5) equals $x^{\nu+\mu}\Gamma(\mu)$.

Thus, the equality (5) takes the form:

$$\int_a^x x^{-\nu} f(u)du = f^*(x), \tag{6}$$

$$f^*(x) = 2^{1-\mu}\frac{x^{-\nu-\mu}}{\Gamma(\mu)}\int_a^x (x^2-t^2)^{\mu-1} F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{(x^2-t^2)}{x^2}\right) t\, g(t)dt.$$

Differentiating both sides of the (6), we come at the following form of the equation solution (1):

$$f(x) = x^\nu \frac{d}{dx}\left\{2^{1-\mu}\frac{x^{-\nu-\mu}}{\Gamma(\mu)}\int_a^x (x^2-t^2)^{\mu-1} F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{x^2-t^2}{x^2}\right) t\, g(t)dt\right\}. \tag{7}$$

Thus, we have proved that if the equation (1) is solvable, then its solution has the form (7).

Necessary and sufficient solvability conditions.

Prove the theorem which gives necessary and sufficient conditions of solvability of equation (1) in the space $L_1\left([a,b],t^{-\nu}\right)=\left\{f(t): t^{-\nu} f(t)\in L_1(a,b)\right\}$, where $L_1(a,b)=\left\{f(x):\int_a^b |f(t)|dt<\infty\right\}$, in terms of auxiliary function $f^*(x)$. In the proof we use the fact that the space of absolutely continuous functions coincides with the class of primitive from the Lebesgue integrable functions [3, p. 338; 4, p. 368-369]:

$$g(x)\in AC([a,b]) \Leftrightarrow g(x) = c+\int_a^x f(t)dt,\ \int_a^b |f(t)|dt<\infty,$$

therefore, absolutely continuous functions are integrable almost everywhere derivative $g'(x)$.

Theorem 1. *The Abel-type integral equation (1) with* $\mu, \nu, 0<\mu<1$, *is solvable in the space* $L_1\left([a,b],t^{-\nu}\right)$ *if and only if*

$$f^*(x) = 2^{1-\mu}\frac{x^{-\nu-\mu}}{\Gamma(\mu)}\int_a^x (x^2-t^2)^{\mu-1}F\left(\frac{\nu}{2}+\frac{\mu}{2},\frac{1}{2}-\frac{\nu}{2}+\frac{\mu}{2};\mu;\frac{(x^2-t^2)}{x^2}\right)t\,g(t)dt \in AC([a,b]),$$

and $f^*(a)=0$.

Under these conditions, equation (1) is uniquely solvable in $L_1\left([a,b],t^{-\nu}\right)$ *and its solution is given by (7).*

REFERENCES

1. Higher transcendental functions (Bateman Manuscript Project) / A. Erdelyi [et al.] // McGraw-Hill., New York, 1953 ; Nauka, Moscow, 1965. – Vol. 1.
2. Самко, С.Г. Интегралы и производные дробного порядка и некоторые их приложения / С.Г. Самко, А.А. Килбас, О.И. Маричев. – Минск : Наука и техника, 1987. – 688 с.
3. Колмогоров, А.Н. Элементы теории функций и функционального анализа / А.Н. Колмогоров, С.В. Фомин. – М. : Наука, 1968. – 496 с.
4. Никольский, С.М. Курс математического анализа / С.М. Никольский. – М. : Наука, 1983. – Т. 2. – 448 с.