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INTEGRAL TRANSFORM WITH THE LEGENDRE FUNCTION OF THE FIRST KIND IN THE KERNEL ON $L_{\nu,r}$ -SPACE

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This paper is devoted to the study of one class of one-dimensional integral transform

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(x) = \int_0^x \left(x^2 - t^2\right)^{-\gamma/2} P^{\gamma}_{\delta}\left(\frac{x}{t}\right) f(t)dt \quad (x > 0),$$

involving the Legendre function of the first kind $P^{\gamma}_{\delta}(z)$ *with complex* $\gamma, \delta, \mathrm{Re}(\gamma) < 1$, *in the kernel, in the space* $\mathfrak{L}_{\nu,r}$ *of Lebesgue measurable functions* f *on* $R_+ = (0,\infty)$ *such that* $\left(\int_0^\infty \left|t^{\nu} f(t)\right|^r \frac{dt}{t}\right)^{\frac{1}{r}}$ $(1 < r < \infty, \nu \in R)$. *Mapping properties such as the boundedness, the range, the representation and the inversion of the considered transforms are established.*

1. Introduction.

We consider the integral transform

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(x) = \int_0^x \left(x^2 - t^2\right)^{-\gamma/2} P^{\gamma}_{\delta}\left(\frac{x}{t}\right) f(t)dt \quad (x > 0), \tag{1}$$

containing the Legendre function of the first kind $P^{\gamma}_{\delta}(z)$ in the kernel, see [1, formulas 3.2(3) and 3.4(6)] and [2, Section 2.1].

Our paper is devoted to the study of transform $\mathrm{P}^{\gamma}_{\delta,1} f$ in the weighted spaces of Lebesgue measurable, generally speaking complex, functions f on $R_+ = (0,\infty)$ such that $\|f\|_{\nu,r} < \infty$, where

$$\|f\|_{\nu,r} = \left(\int_0^\infty \left|t^{\nu} f(t)\right|^r \frac{dt}{t}\right)^{\frac{1}{r}} \quad (1 < r < \infty, \nu \in R). \tag{2}$$

Our investigations are based on representation of equation (1) via the so-called modified H- transform of the form

$$\left(H^{1}_{\sigma,\kappa} f\right)(x) = x^{\sigma} \int_0^\infty H^{m,n}_{p,q}\left[\frac{x}{t}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] t^{\kappa} f(t)\frac{dt}{t} \quad (x > 0) \tag{3}$$

with the H – function $H^{m,n}_{p,q}[z]$ in the kernel. Such a function is defined for integral nonnegative m,n,p,q $(0 \le m \le q, 0 \le n \le p)$, for complex $a_i, b_j \in C$ and positive $\alpha_i > 0, \beta_j > 0$ $(i = 1,..,p;\ j = 1,..,q)$ by

$$H^{m,n}_{p,q}[z] \equiv H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \mathcal{H}^{m,n}_{p,q}(s) z^{-s} ds,\ z \ne 0, \tag{4}$$

where

$$\mathcal{H}^{m,n}_{p,q}(s) \equiv \mathcal{H}^{m,n}_{p,q}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right| s\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right] = \frac{\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\prod_{i=1}^{n}\Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i + \alpha_i s)\prod_{j=m+1}^{q}\Gamma(1 - b_j - \beta_j s)}. \tag{5}$$

Here L is a specially chosen infinite contour and an empty product, if it occurs, being taken to be one [2, Section 8.3], [3, Chapters 1 and 2], [4].

In this paper we apply results from [3, Chapters 5.5] to prove the boundedness and one-to-one property of the operator $\mathrm{P}^{\gamma}_{\delta,1} f$ of transform (1) from one space $L_{\nu,r}$ to the another, to present various integral representation and characterization of images of these operators, and to establish their inversion formulae.

2. Representation in the form of modified H-transform.

The Mellin transform formula of the transform $\mathrm{P}^{\gamma}_{\delta,1} f$ was obtained for suitable functions f in the [4, formula 45] and has the form

$$\left(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,1} f\right)(s) = 2^{\gamma-1} \frac{\Gamma\left((1+\gamma+\delta-s)/2\right)\Gamma\left((\gamma-\delta-s)/2\right)}{\Gamma\left(1-(s/2)\right)\Gamma\left((1-s)/2\right)} \left(\mathfrak{M}f\right)(1-\gamma+s). \quad (6)$$

In accordance with (5), relation (6) takes the form

$$\left(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,1} f\right)(s) = 2^{\gamma-1} \frac{\Gamma\left((1+\gamma+\delta-s)/2\right)\Gamma\left((\gamma-\delta-s)/2\right)}{\Gamma\left(1-(s/2)\right)\Gamma\left((1-s)/2\right)} \left(\mathfrak{M}f\right)(1-\gamma+s) =$$

$$= 2^{\gamma-1} \mathcal{H}^{0,2}_{2,2}\left[\left.\begin{matrix} \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right) & \left(1+\frac{\delta-\gamma}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right| s\right] \left(\mathfrak{M}f\right)(1-\gamma+s). \quad (7)$$

Therefore, by [4, (13)], the initial integral transform (1) are modified H transform (3), with $\sigma = 0$ and $\kappa = 1-\gamma$:

$$(\mathrm{P}^{\gamma}_{\delta,1} f)(x) = 2^{\gamma-1} \int_0^{\infty} H^{0,2}_{2,2}\left[\frac{x}{t}\left|\begin{matrix} \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right), \left(1+\frac{\delta-\gamma}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right.\right] t^{-\gamma} f(t)dt. \quad (8)$$

3. $L_{\nu,2}$ - theory of transform $\mathrm{P}^{\gamma}_{\delta,1} f$.

$L_{\nu,2}$ - theory of transform follows from equation (8) with using Proposition 1 for the $H^1_{\sigma,\kappa}$ in [4].

Theorem 1 Let $-\infty < \nu - \mathrm{Re}(1-\gamma) < \min[\mathrm{Re}(1+\gamma+\delta), \mathrm{Re}(\gamma-\delta)]$, $\mathrm{Re}(\gamma) \le 1$. (9)

There hold the following assertions:

a) There exists a one-to-one map $\mathrm{P}^{\gamma}_{\delta,1} \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{\nu-\mathrm{Re}(1-\gamma),2}]$ such that relation (7) holds for $f \in L_{\nu,2}$ и $\mathrm{Re}(s) = \nu - \mathrm{Re}(1-\gamma)$. If $\mathrm{Re}(\gamma) = 1$ and $s \neq 2m+1$, $s \neq 2l+2$ $(l, m \in N_0 = N \bigcup \{0\})$ for $\mathrm{Re}(s) = 1-\nu$ holds, then $\mathrm{P}^{\gamma}_{\delta,1}$ is one-to-one on $L_{\nu,2}$.

b) The transform $\mathrm{P}^{\gamma}_{\delta,1} f$ does not depend on ν in the sense if ν and $\tilde{\nu}$ satisfy equation (9) and if the transforms $\mathrm{P}^{\gamma}_{\delta,1} f$ and $\tilde{\mathrm{P}}^{\gamma}_{\delta,1} f$ are defined in respective spaces $L_{\nu,2}$ and $L_{\tilde{\nu},2}$ by equation (7), then $\mathrm{P}^{\gamma}_{\delta,1} f = \tilde{\mathrm{P}}^{\gamma}_{\delta,1} f$ for $f \in L_{\tilde{\nu},2} \cap L_{\nu,2}$.

c) If $\mathrm{Re}(\gamma) < 1$, then for $f \in L_{\nu,2}$ $\mathrm{P}^{\gamma}_{\delta,1} f$ is given by equations (1) and (8).

d) Let $\lambda \in C, h > 0$, and $f \in L_{\nu,2}$. If $\mathrm{Re}(\lambda) > (\nu - \mathrm{Re}(1-\gamma))h - 1$, then $\mathrm{P}^{\gamma}_{\delta,1} f$ is represented in the form

$$(\mathrm{P}^{\gamma}_{\delta,1} f)(x) = 2^{\gamma-1} h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^{\infty} H^{0,3}_{3,3}\left[\frac{x}{t}\left|\begin{matrix} (-\lambda, h), \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right), \left(1+\frac{\delta-\gamma}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (-\lambda-1, h) \end{matrix}\right.\right] t^{-\gamma} f(t)dt, \quad (10)$$

while for $\mathrm{Re}(\lambda) < (\nu - \mathrm{Re}(1-\gamma))h - 1$ is given by

$$(\mathrm{P}^{\gamma}_{\delta,1} f)(x) = -2^{\gamma-1} h x^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h} \int_0^{\infty} H^{1,2}_{3,3}\left[\frac{x}{t}\left|\begin{matrix} \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right), \left(1+\frac{\delta-\gamma}{2}, \frac{1}{2}\right), (-\lambda, h) \\ (-\lambda-1, h), \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (-\lambda-1, h) \end{matrix}\right.\right] t^{-\gamma} f(t)dt. \quad (11)$$

e) If $f \in L_{\nu,2}$ and $g \in L_{1-\nu+\mathrm{Re}(1-\gamma),2}$, then there holds the relation

$$\int_0^\infty f(x)\left(\mathrm{P}^{\gamma}_{\delta,1}g\right)(x)dx = \int_0^\infty 2^{\gamma-1}\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(x)g(x)dx\,, \qquad (12)$$

where $\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(x)$ is the transform $\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(x) = \int_x^\infty \left(t^2 - x^2\right)^{-\gamma/2} P^{\gamma}_{\delta}\left(\frac{t}{x}\right) f(t)dt \quad (x > 0)$ [4, (2)].

4. $L_{\nu,r}$ - theory of transform $\mathrm{P}^{\gamma}_{\delta,1}f$.

Acccording to (1), from Proposition 2 and Proposition 3 in [4] for the $H^{1}_{\sigma,\kappa}$ - transform we deduce the results for the transform $\mathrm{P}^{\gamma}_{\delta,1}f$.

Theorem 2 Let $-\infty < \nu < \min[\mathrm{Re}(2+\delta), \mathrm{Re}(1-\delta)]$ and $1 < r < \infty$.

There the following assertions are true:

a) The transform $\mathrm{P}^{\gamma}_{\delta,1}f$, defined on $L_{\nu,2}$, can be extended to $L_{\nu,r}$. If $1 < r \le 2$, then the transform $\mathrm{P}^{\gamma}_{\delta,1}f$ is one-to-one and there holds the relation (7) for $f \in \mathfrak{L}_{\nu,r}$ and $\mathrm{Re}(s) = \nu$.

b) If $f \in L_{\nu,2}$, $\lambda \in C$ and $h > 0$, then $\mathrm{P}^{\gamma}_{\delta,1}f$ is represented in the form (10) for $\mathrm{Re}(\lambda) > \nu h - 1$ and in the form (11) for $\mathrm{Re}(\lambda) < \nu h - 1$.

c) If $f \in L_{\nu,r}$ and $g \in L_{1-\nu,r'}$, where $r' = r/(r-1)$, then the formula (12) holds.

d) If $s \ne 2m+1, \quad s \ne 2l+2 \quad (l, m \in N_0 = N \cup \{0\})$ for $\mathrm{Re}(s) = 1-\nu$, then $\mathrm{P}^{\gamma}_{\delta,1}$ is one-to-one on $L_{\nu,r}$ and its image is given by

$$\mathrm{P}^{\gamma}_{\delta,1}f\ (\mathfrak{L}_{\nu,r}) = \mathfrak{L}_{\nu,r}\ . \qquad (13)$$

Theorem 3 Let $\mathrm{Re}(\gamma) < 1$, $-\infty < \nu - \mathrm{Re}(1-\gamma) < \min[\mathrm{Re}(1+\gamma+\delta), \mathrm{Re}(\gamma-\delta)]$, $1 < r < \infty$.

There the following assertions are true:

a) The transform $\mathrm{P}^{\gamma}_{\delta,1}f$, defined on $L_{\nu,2}$, can be extended to $L_{\nu,r}$ as an element of $P^{\gamma}_{\delta,1} \in [\,L_{\nu,r}\,; L_{\nu-\mathrm{Re}(1-\gamma),s}\,]$ for any $s \ge r$ such that $\frac{1}{s} > \frac{1}{r} + \mathrm{Re}(\gamma-1)$. If $1 < r \le 2$, then $\mathrm{P}^{\gamma}_{\delta,1}f$ is a one-to-one and there holds the relation (7) for $f \in L_{\nu,r}$ and $\mathrm{Re}(s) = \nu - \mathrm{Re}(1-\gamma)$.

b) For $f \in L_{\nu,r}$, the transform $\mathrm{P}^{\gamma}_{\delta,1}f$ is given by equation (8).

c) If $f \in L_{\nu,r}$, $\lambda \in C$ and $h > 0$, then $\mathrm{P}^{\gamma}_{\delta,1}f$ is represented by (10) for $\mathrm{Re}(\lambda) > (\nu - \mathrm{Re}(1-\gamma))h - 1$, while by equation (11) for $\mathrm{Re}(\lambda) < (\nu - \mathrm{Re}(1-\gamma))h - 1$.

d) If $f \in L_{\nu,r}$ and $g \in L_{1-\nu+\mathrm{Re}(1-\gamma),s}$, where где $1 < s < \infty, 1 \le 1/r + 1/s < 1 - \mathrm{Re}(\gamma-1)$, then the relation (12) holds.

e) If $s \ne 2m+1, \quad s \ne 2l+2 \quad (l, m \in N_0 = N \cup \{0\})$ for $\mathrm{Re}(s) = 1-\nu$, then $\mathrm{P}^{\gamma}_{\delta,1}f$ is a one-to-one on $L_{\nu,r}$ and its image is characterized by

$$\mathrm{P}^{\gamma}_{\delta,1}\ (L_{\nu,r}) = \mathrm{I}^{1-\gamma}_{0+;1-\gamma,\beta/(-\gamma)}\ (L_{\nu-\mathrm{Re}(1-\gamma),r})\,, \qquad (14)$$

where $\beta = \min[\mathrm{Re}(1+\gamma+\delta), \mathrm{Re}(\gamma-\delta)]$. If $\mathrm{Re}(s) \ne 1-\nu+\mathrm{Re}(1-\gamma)$, then $\mathrm{P}^{\gamma}_{\delta,1}\ (L_{\nu,r})$ is a subset of the right-hand side of equation (14).

5. Inversion formulas of the transform $\mathrm{P}^{\gamma}_{\delta,1}f$.

According to (8) the relation formulas (31) and (32) from [4] for $\mathrm{P}^{\gamma}_{\delta,1}f$ take the forms

$$f(x) = -2^{1-\gamma}hx^{(\lambda+1)/h-1+\gamma}\frac{d}{dx}x^{-(\lambda+1)/h}\int_0^\infty H^{2,1}_{3,3}\left[\frac{t}{x}\left|\begin{matrix}(-\lambda,h),\left(\frac{\gamma+\delta}{2},\frac{1}{2}\right),\left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right)\\ \left(\frac{1}{2},\frac{1}{2}\right),\left(0,\frac{1}{2}\right),(-\lambda-1,h)\end{matrix}\right.\right](\mathrm{P}^{\gamma}_{\delta,1}f)(t)dt \qquad (15)$$

or

$$f(x) = 2^{1-\gamma} h x^{(\lambda+1)/h-1} \frac{d}{dx} x^{-(\lambda+1)/h} \int_0^\infty H_{3,3}^{3,0} \left[\frac{t}{x} \left| \begin{matrix} \left(\frac{\gamma+\delta}{2}, \frac{1}{2}\right), \left(\frac{\gamma-\delta-1}{2}, \frac{1}{2}\right), (-\lambda, h) \\ (-\lambda-1, h), \left(\frac{1}{2}, \frac{1}{2}\right), \left(0, \frac{1}{2}\right) \end{matrix} \right. \right] (\mathrm{P}_{\delta,1}^{\gamma} f)(t) dt . \tag{16}$$

Conditions for the validity of formulas (15) and (16) are followed and from Proposition 4 in [4].

Theorem 3 Let

$$-\infty < \nu < \min[1, \mathrm{Re}(2+\delta), \mathrm{Re}(1-\delta)] \text{ and let } \lambda \in C,\ h > 0 .$$

a) If $f \in \mathfrak{L}_{\nu,2}$, then the inversion formulas (15) and (16) are valid for $\mathrm{Re}(\lambda) > (1-\nu)h-1$ and $\mathrm{Re}(\lambda) < (1-\nu)h-1$, respectively.

b) If $f \in \mathfrak{L}_{\nu,r}$, $1 < r < \infty$, then the relations (15) and (16) are true in the respective cases $\mathrm{Re}(\lambda) > (1-\nu)h-1$ and $\mathrm{Re}(\lambda) < (1-\nu)h-1$.

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