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ON THE OSCILLATION AND ASYMPTOTIC BEHAVIOR FOR A HIGER ORDER NEUTRAL DIFFERENCE EQUATION

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In this paper, the oscillation and asymptotic behavior of the higer order neutral difference equation $\Delta^k(x_n+\delta_n x_{n-\tau})+\sum_{i=1}^{r}\alpha_i(n)F(x_{n-m_i})=0,\quad n=0,1,\cdots$ *are investigated.*

1. Introduction. The properties of solutions of neutral difference equations has been studied extensively in recent years; (see for example the work in [1 – 10] and the references cited therein). In [3], we obtained some results for the oscillation and the convergence of solutions of neutral difference equation of the form

$$\Delta(x_n+\delta x_{n-\tau})+\sum_{i=1}^{r}\alpha_i(n)F(x_{n-m_i})=0, \tag{1}$$

for $n\in\mathbb{N}$, $n\geq n_0$ for some $n_0\in\mathbb{N}$, where $r, m_1, m_2, \cdots, m_r$ are fixed positive integers, the functions $\alpha_i(n)$ are defined on $\mathbb{N}$ and the function F is defined on $\mathbb{R}$. In [1], the author obtained some results for the oscillation and the convergence of solutions of higer order neutral difference equation of the form

$$\Delta^k(x_n+\delta_n x_{n-\tau})+q_nF(x_{n-\sigma})=0 \tag{2}$$

with some restrictions on the function F, the sequences $(q_n),(\delta_n)$.

Motivated by the work above, in this paper, we aim to study the oscillation and convergence of solutions of higer order neutral difference equation

$$\Delta^k(x_n+\delta_n x_{n-\tau})+\sum_{i=1}^{r}\alpha_i(n)F(x_{n-m_i})=0, \tag{3}$$

for $n\in\mathbb{N}$, where $k, \tau, r, m_1, m_2, \cdots, m_r$ are fixed positive integers and the functions $\alpha_i(n)$ are defined on $\mathbb{N}$, $\alpha_i(n)\geq 0$, and are not eventually identically zero, the continuous function $F:\mathbb{R}\to\mathbb{R}$ is such that $xF(x)>0$ for all $x\neq 0$. Moreover, with respect to (3), we assume that there exists a function $G:\mathbb{R}\to\mathbb{R}$ such that G is continuous and nondecreasing and satisfies the inequality

$$G(xy)\geq MG(x)G(y)\quad \text{for } x, y>0,$$

where M is a positive constant,

$$|F(x)|\geq|G(x)|,\quad \frac{G(x)}{x}\geq N>0$$

and $xG(x)>0$ for $x\neq 0$.

Put $A=\max\{\tau, m_1, \cdots, m_r\}$. Then, by a solution of (3) we mean a function which is defined for $n\geq -A$ and sastisfies the equation (3) for $n\in\mathbb{N}$. Clearly, if

$$x_n=a_n,\quad n=-A,-A+1,\cdots,-1,0$$

are given, then (3) has a unique solution, and it can be constructed recursively.

A nontrivial solution $(x_n)_{n\geq n_0}$ of (3) is called *oscillatory* if for any $n_1\geq n_0$ there exists $n_2\geq n_1$ such that $x_{n_2}x_{n_2+1}\leq 0$. The difference equation (3) is called oscillatory if all its solutions are oscillatory. Otherwise, it is called nonoscillatory.

2. The results. To begin with, we get theorem following.

THEOREM 2.1 [2] (Discrete Kneser's Theorem). *Let $(x_n)_{n\ge n_0}$ be such that $x_n > 0$ with $\Delta^k x_n$ of constant sign for all $n \in \mathbb{N}$, $n \ge n_0$ and not identically zero. Then, there exists an integer m, $0 \le m \le k$ with $k+m$ odd for $\Delta^k x_n \le 0$ or $k+m$ even for $\Delta^k x_n \ge 0$ and such that*:

$m \le k-1$ implies $(-1)^{m+i}\Delta^i x_n > 0$ for all $n \in \mathbb{N}$, $n \ge n_0$, $m \le i \le n-1$;

$m \ge 1$ implies $(-1)^{m+i}\Delta^i x_n > 0$ for all $n \in \mathbb{N}$, $n \ge n_0$, $1 \le i \le m-1$.

Corollaly 2.2 [2]. *Let $(x_n)_{n\ge n_0}$ be such that $x_n > 0$ with $\Delta^k x_n \le 0$ for all $n \in \mathbb{N}$, $n \ge n_0$ and not identically zero. Then, there exists a large integer $n_1 \ge n_0$ such that for all $n \ge n_1$*

$$x_n \ge \frac{1}{(k-1)!}\Delta^{k-1} x_{2^{k-m-1}n}(n-n_1)^{(k-1)}.$$

THEOREM 2.3. *Assume that*

$$\frac{(\tilde m+1)^{\tilde m+1}}{\tilde m^{\tilde m}}\sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n) > 1, \tag{4}$$

where $\alpha_i(n) \ge 0, n \in \mathbb{N}, 1 \le i \le r$ and $\tilde m = \min_{1\le i\le r} m_i$. Then, the inequality

$$\Delta x_n + \sum_{i=1}^{r}\alpha_i(n)x_{n-m_i} \le 0, \quad n \in \mathbb{N}$$

has no eventually positive solution.

Proof. Assume, for the sake of contradiction, that (4) has a solution (x_n) with $x_n > 0$ for all $n \ge n_1$, $n_1 \in \mathbb{N}$. Setting $v_n = \dfrac{x_n}{x_{n+1}}$ and dividing this inequality by x_n, we obtain

$$\frac{1}{v_n} \le 1 - \sum_{i=1}^{r}\alpha_i(n)\prod_{\ell=1}^{m_i} v_{n-\ell}, \tag{5}$$

where $n \ge n_1 + m, \quad m = \max_{1\le i\le r} m_i$.

Clearly, (x_n) is nonincreasing with $n \ge n_1 + m$, and so $v_n \ge 1$ for all $n \ge n_1 + m$. From (4) and (5) we see that (v_n) is a above bounded sequence. Putting $\liminf_{n\to\infty} v_n = \beta$, we get

$$\limsup_{n\to\infty}\frac{1}{v_n} = \frac{1}{\beta} \le 1 - \liminf_{n\to\infty}\sum_{i=1}^{r}\alpha_i(n)\prod_{\ell=1}^{m_i} v_{n-\ell},$$

or

$$\frac{1}{\beta} \le 1 - \sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\cdot\beta^{m_i}. \tag{6}$$

Since

$$\beta^{m_i} \ge \beta^{\tilde m}, \quad \forall i = \overline{1,r},$$

we have

$$\liminf_{n\to\infty}\alpha_i(n)\beta^{m_i} \ge \liminf_{n\to\infty}\alpha_i(n)\beta^{\tilde m}, \quad \forall i = \overline{1,r}$$

and

$$1 - \sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\beta^{m_i} \le 1 - \sum_{i=1}^{r}\liminf_{n\to\infty}\alpha_i(n)\beta^{\tilde m}.$$

From (6) we have

$$\liminf_{n\to\infty}\sum_{i=1}^{r}\alpha_i(n) \le \frac{\beta-1}{\beta^{\tilde m+1}}.$$

But

$$\frac{\beta-1}{\beta^{\tilde{m}+1}} \le \frac{\tilde{m}^{\tilde{m}}}{(\tilde{m}+1)^{\tilde{m}+1}},$$

so

$$\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} \sum_{i=1}^{r} \lim_{n\to\infty} \inf \alpha_i(n) \le 1,$$

which contradicts condition (4). Hence, the inequality

$$\Delta x_n + \sum_{i=1}^{r} \alpha_i(n) x_{n-m_i} \le 0, \quad n \in \mathbb{N}$$

has no eventually positive solution. The proof is complete.

THEOREM 2.4. *Let* k *be even. Assume that* $0 \le \delta_n < 1, \quad n \ge n_0$ *and*

$$\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} M^2 NG\left(\frac{1}{(k-1)!}\right) \sum_{i=1}^{r} \lim_{n\to\infty} \inf \alpha_i(n) G\left(1-\delta_{n-m_i}\left(\frac{n-m_i}{2^{n-1}}\right)^{(k-1)}\right) > 1, \tag{7}$$

where $\tilde{m} = \min_{1\le i\le r} m_i$. Then, the equation (3) is oscillatory.

Proof. Let (x_n) be a nonoscillatory solution of (3) with $x_n > 0, x_{n-\tau} > 0$ and $x_{n-m_i} > 0$ for all $n \ge n_0 \ge N_0$ and $i = 1, 2, \cdots, r$. Setting $z_n = x_n + \delta_n x_{n-\tau}$, we get $z_n \ge x_n > 0$ and

$$\Delta^k z_n = -\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) < 0, \quad n \ge n_0. \tag{8}$$

It follows from Theorem 2.1 that

$$\Delta^{k-1} z_n > 0, \quad k \ge 2, n \ge n_0. \tag{9}$$

We will prove that $\Delta z_n < 0$ eventually. This is obvious from the equation (3) in the case $k = 1$. For $k \ge 2$, we suppose on the contrary that $\Delta z_n > 0$ for $n \ge n_1 \ge n_0$. Then

$$(1-\delta_n) z_n \le z_n - \delta_n z_{n-\tau} = x_n - \delta_n \delta_{n-\tau} x_{n-2\tau} \le x_n \tag{10}$$

for $n \ge n_2 \ge n_1$. Since (z_n) is positive and increasing, it follows from Corollary 2.2 and (10) that

$$x_n \ge (1-\delta_n) z_n \ge \frac{1-\delta_n}{(k-1)!}\left(\frac{n}{2^{k-1}}\right)^{(k-1)} \Delta^{k-1} z_n, \quad n \ge 2^{k-1} n_2. \tag{11}$$

From (11) for $n \ge n_3 \ge n_2$, we obtain

$$\begin{aligned} F(x_{n-m_i}) &\ge G(x_{n-m_i}) \\ &\ge G\left(\frac{1-\delta_{n-m_i}}{(k-1)!}\left(\frac{n-m_i}{2^{k-1}}\right)^{(k-1)} \Delta^{k-1} z_{n-m_i}\right) \\ &\ge M^2 NG\left(\frac{1}{(k-1)!}\right) G\left((1-\delta_{n-m_i})\left(\frac{n-m_i}{2^{k-1}}\right)^{(k-1)}\right) \Delta^{k-1} z_{n-m_i}. \end{aligned}$$

Put $w_n = \Delta^{k-1} z_n, \quad n \ge n_0$. From (8) we have

$$\Delta w_n + \sum_{i=1}^{r} \alpha_i(n) M^2 NG\left(\frac{1}{(k-1)!}\right) G\left((1-\delta_{n-m_i})\left(\frac{n-m_i}{2^{k-1}}\right)^{(k-1)}\right) w_{n-m_i} \le 0. \tag{12}$$

We see that (w_n) is an eventually positive solution of (12). But, in view of the condition (7), this is a contradiction to Theorem 2.3. Hence, $\Delta z_n < 0$ eventually.

Since $\Delta z_n < 0$ eventually, in Theorem 2.1 we must have $m = j = 0$, and

$$(-1)^i \Delta^i z_n > 0, \quad 0 \le i \le k-1, \quad n \ge n_0. \tag{13}$$

If k is even, (13) implies a contradiction to (9). The proof is complete.

THEOREM 2.5. *Let* k *be odd. Assume that* $0 \le \delta_n \le \sigma < 1, \quad n \ge n_0$ *where* σ *is a constant and*

$$\frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} M^2 NG(P) \sum_{i=1}^{r} \liminf_{n\to\infty} \alpha_i(n) G\left(\left(\frac{n-m_i}{2^{k-1}} \right)^{(k-1)} \right) > 1, \tag{14}$$

for every $P \in (0,1)$, $\tilde{m} = \min_{1 \le i \le r} m_i$. Then, every solution of (3) either oscillates or tends to zero as $n \to \infty$.

Proof. Assume that (x_n) does not tend to zero as $n \to \infty$. Using proceeding as in the proof of Theorem 2.4, we have $\Delta z_n < 0$ eventually. This implies that $z_n \to \ell$ as $n \to \infty$, where $0 < \ell < \infty$. Then, there exists $\varepsilon > 0$ and an integer $n_4 > n_0$ such that

$$0 < \varepsilon < \ell \frac{1-\sigma}{1+\sigma} < \ell$$

and

$$\ell - \varepsilon < z_n \le z_{n-\tau} < \ell + \varepsilon, \quad n \le n_4. \tag{15}$$

Thus, from (10) and (15), we find for $n \ge n_4$ that

$$x_n \ge z_n - \delta_n z_{n-\tau} \ge z_n - \sigma z_{n-\tau} > \ell - \varepsilon - \sigma(\ell+\varepsilon) > \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} z_n.$$

Let $m = j$ be as in Corollary 2.2. We have

$$z_n = \frac{z_n}{z_{2^{j+1-k}n}} z_{2^{j+1-k}n} > \frac{\ell-\varepsilon}{\ell+\varepsilon} z_{2^{j+1-k}n}, \quad n \ge n_5 > n_4. \tag{16}$$

Combining (15) and (16) and using Corollary 2.2, we get for $n \ge n_6 > n_5$ that

$$\begin{aligned} x_n &> \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} z_{2^{j+1-k}n} \\ &\ge \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} \frac{(2^{j+1-k}n - n_6)^{(k-1)}}{(k-1)!} \Delta^{k-1} z_n \\ &\ge \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} \cdot \frac{1}{(k-1)!} 2^{(j+1-k)(k-1)} (n - 2^k n_6)^{(k-1)} \Delta^{k-1} z_n. \end{aligned}$$

Thus, for $n \ge 2^{k+1} n_6 + k - 2$ it follows that

$$\begin{aligned} x_n &\ge \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} \cdot \frac{1}{(k-1)!} 2^{(j+1-k)(k-1)} \frac{1}{2^{k-1}} (n)^{(k-1)} \Delta^{k-1} z_n \\ &\ge \frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} \cdot \frac{1}{(k-1)!} 2^{(j-k)(k-1)} \Delta^{k-1} z_n. \end{aligned} \tag{17}$$

It can easily be seen that $\frac{\ell - \varepsilon - \sigma(\ell+\varepsilon)}{\ell+\varepsilon} \cdot \frac{\ell-\varepsilon}{\ell+\varepsilon} \cdot \frac{1}{(k-1)!} 2^{(j-k)(k-1)} = P \in (0,1)$.

By (17), for $n \ge n_7 > n_6$, we obtain

$$\sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \ge \sum_{i=1}^{r} \alpha_i(n) M^2 NG(P) G\left(\left(\frac{n-m_i}{2^{k-1}} \right)^{(k-1)} \right) \Delta^{k-1} z_{n-m_i}.$$

Put $w_n = \Delta^{k-1} z_n, \quad n \ge n_0$. We see that (w_n) is an eventually positive solution of

$$\Delta w_n + \sum_{i=1}^{r} \alpha_i(n) M^2 NG(P) G\left(\left(\frac{n-m_i}{2^{k-1}} \right)^{(k-1)} \right) w_{n-m_i} \le 0.$$

In view of the conditon (14), this is a contradiction to Theorem 2.3. The proof is complete.

THEOREM 2.6. *Assume that* $-1<-\sigma\le\delta_n\le 0,\quad n\ge n_0$ *where* σ *is a constant and the condition (14) in Theorem 2.5 is satisfied. Then, every solution of (3) either oscillates or tends to zero as* $n\to\infty$.

Proof. Let (x_n) be a nonoscillatory solution of (3) with $x_n>0, x_{n-\tau}>0$ and $x_{n-m_i}>0$ for all $n\ge n_0\ge N_0$ and $i=1,2,\cdots,r$. Assume, furthermore, that (x_n) does not tend to zero as $n\to\infty$. Setting $z_n=x_n+\delta_n x_{n-\tau}$, we get $z_n\le x_n$ and

$$\Delta^k z_n=-\sum_{i=1}^{r}\alpha_i(n)F(x_{n-m_i})<0,\quad n\ge n_0. \tag{18}$$

We claim that $\Delta x_n\le 0$ eventually. Suppose on the contrary that $\Delta z_n>0$ for $n\ge n_1>n_0$. Then, for $n\ge n_2>n_1$, we have

$$z_n\ge x_n+\delta_n x_n\ge(1-\sigma)x_n>0. \tag{19}$$

Thus, inequality (9) follows from Theorem 2.1. Since (x_n) is unbounded, it follows from (19) that (z_n) is also unbounded, and hence $\Delta z_n>0,\quad n\ge n_2$. Applying Corollary 2.2, we find

$$x_n\ge z_n\ge\frac{1}{(k-1)!}\left(\frac{n}{2^{k-1}}\right)^{(k-1)}\Delta^{k-1}z_n,\quad n\ge 2^{k-1}n_2. \tag{20}$$

Therefore, in view of (20), for $n\ge n_3>n_2$ we obtain

$$\begin{aligned} F(x_{n-m_i}) &\ge G(x_{n-m_i})\\ &\ge M^2NG\left(\left(\frac{1}{(k-1)!}\right)G\left(\frac{n-m_i}{2^{k-1}}\right)^{(k-1)}\right)\Delta^{k-1}z_{n-m_i}. \end{aligned}$$

It follows from (9) and the above inequality that $\Delta^{k-1}z_n$ is an eventually positive solution of

$$\Delta w_n+\sum_{i=1}^{r}\alpha_i(n)M^2NG\left(\frac{1}{(k-1)!}\right)G\left(\left(\frac{n-m_i}{2^{k-1}}\right)^{(k-1)}\right)w_{n-m_i}\le 0.$$

In view of the conditon (14), this is a contradiction to Theorem 2.3. Hence, $\Delta x_n\le 0$ eventually. This implies that $x_n\to\ell$ as $n\to\infty$, where $0<\ell<\infty$.

Since $z_n=x_n+\delta_n x_{n-\tau}$, we get

$$\liminf_{n\to\infty} z_n=(1+\liminf_{n\to\infty}\delta_n)\ell\ge(1-\sigma)\ell.$$

Hence, (z_n) is eventually positive and (9) holds. Since $z_n\le x_n$ and (x_n) is nonincreasing eventually, (z_n) is also nonincreasing eventually. Thus, $z_n\to\ell_1$ as $n\to\infty$, where $0<\ell_1<\infty$. Given $\varepsilon\in(0,\ell_1)$, there exists an integer $n_4>n_0$ such that

$$\ell_1-\varepsilon<z_n<\ell_1+\varepsilon,\quad n\ge n_4. \tag{21}$$

Let $m=j$ be as in Corollary 2.2. For $n\ge n_5>n_4$, using (21) and Corollary 2.2 successively, we obtain

$$\begin{aligned} z_n &= \frac{z_n}{z_{2^{j+1-k}n}}z_{2^{j+1-k}n}\\ &> \frac{\ell_1-\varepsilon}{\ell_1+\varepsilon}z_{2^{j+1-k}n}\\ &\ge \frac{\ell_1-\varepsilon}{\ell_1+\varepsilon}\frac{(2^{j+1-k}n-n_5)^{(k-1)}}{(k-1)!}\Delta^{k-1}z_n\\ &\ge \frac{\ell_1-\varepsilon}{\ell_1+\varepsilon}\frac{1}{(k-1)!}2^{(j+1-k)(k-1)}(n-2^n n_5)^{(k-1)}\Delta^{k-1}z_n. \end{aligned}$$

It follows that for $n \ge 2^{k+1} n_5 + k - 2$,

$$\begin{aligned} z_n &\ge \frac{\ell_1 - \varepsilon}{\ell_1 + \varepsilon} \cdot \frac{1}{(k-1)!} 2^{(j+1-k)(k-1)} \frac{1}{2^{k-1}} (n)^{(k-1)} \Delta^{k-1} z_n \\ &\ge \frac{2^{(j-k)(k-1)}(\ell_1 - \varepsilon)}{(\ell_1 + \varepsilon)(k-1)!} \cdot \left(\frac{n}{2^{k-1}} \right)^{(k-1)} \Delta^{k-1} z_n. \end{aligned} \quad (22)$$

It is easily seen that $\frac{2^{(j-k)(k-1)}(\ell_1 - \varepsilon)}{(\ell_1 + \varepsilon)(k-1)!} \in (0,1)$. By (22), for $n \ge n_6 > n_5$, we get

$$\begin{aligned} F(x_{n-m_i}) &\ge G(x_{n-m_i}) \ge G(z_{n-m_i}) \\ &\ge M^2 NG\left(\left(\frac{2^{(j-k)(k-1)}(\ell_1 - \varepsilon)}{(\ell_1 + \varepsilon)(k-1)!} \right) G\left(\frac{n - m_i}{2^{k-1}} \right)^{(k-1)} \right) \Delta^{k-1} z_{n-m_i}. \end{aligned}$$

It follows from (9) and the above inequality that $\Delta^{k-1} z_n$ is an eventually positive solution of

$$\Delta w_n + \sum_{i=1}^{r} \alpha_i(n) M^2 NG\left(\frac{2^{(j-k)(k-1)}(\ell_1 - \varepsilon)}{(\ell_1 + \varepsilon)(k-1)!} \right) G\left(\left(\frac{n - m_i}{2^{k-1}} \right)^{(k-1)} \right) w_{n-m_i} \le 0.$$

In view of the conditon (14), this is a contradiction to Theorem 2.3. The proof is complete.

THEOREM 2.7. *Let k be even. Assume that $\delta_n \equiv 1,\ n \ge n_0$ and $\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty$. Then, the equation (3) is oscillatory.*

Proof. Let (x_n) be a nonoscillatory solution of (3) with $x_n > 0, x_{n-\tau} > 0$ and $x_{n-m_i} > 0$ for all $n \ge n_0 \ge N_0$ and $i = 1, 2, \cdots, r$. Setting $z_n = x_n + x_{n-\tau}$, we get $z_n > 0,\ n \ge n_0$ and the inequalities (8) and (9) are satisfied. Summing (3) from n_0 to $n-1$ and using (9), we obtain

$$\Delta^{k-1} z_{n_0} = \sum_{\ell=n_0}^{n-1} \sum_{i=1}^{r} \alpha_i(\ell) F(x_{\ell-m_i}) + \Delta^{k-1} z_n > \sum_{\ell=n_0}^{n-1} \sum_{i=1}^{r} \alpha_i(\ell) N x_{\ell-m_i},$$

which implies

$$\sum_{\ell=n_0}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) x_{\ell-m_i} < \infty. \quad (23)$$

Next, we prove that if $\liminf_{n\to\infty} x_n > 0$, then $\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) < \infty$. Indeed, suppose the contrary that $\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty$. Put $L = \inf_{\ell > n_0} x_{\ell-m_i}, i = 1, 2, \cdots, r$. Then, we have

$$\sum_{\ell=n_0}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) x_{\ell-m_i} \ge L \sum_{\ell=n_0}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty,$$

which contradicts (23).

Since k is even, from Theorem 2.1, we see that $m = j$ is odd and hence $\Delta z_n > 0,\quad n \ge n_0$. Therefore,

$$0 < z_n - z_{n-\tau} = x_n - x_{n-2\tau}, \quad n \ge n_1 > n_0,$$

or $x_n > x_{n-2\tau},\quad n \ge n_1$. This implies $\liminf_{n\to\infty} x_n > 0$. We have seen that this leads to $\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) < \infty$, which is a contradiction to $\sum_{\ell=1}^{\infty} \sum_{i=1}^{r} \alpha_i(\ell) = \infty$. The proof is complete.

THEOREM 2.8. *Let* k *be odd. Assume that* $\delta_n \equiv 1,\ n \geq n_0$ *and* $\sum_{\ell=1}^{\infty}\sum_{i=1}^{r}\alpha_i(\ell) = \infty$. *Then, every solution of (3) either oscillates or tends to zero as* $n \to \infty$.

Proof. Let (x_n) be a nonoscillatory solution of (3) with $x_n > 0,\ x_{n-\tau} > 0$ and $x_{n-m_i} > 0$ for all $n \geq n_0 \geq N_0$ and $i = 1, 2, \cdots, r$. Assume, furthermore, that (x_n) does not tend to zero as $n \to \infty$. From Theorem 2.1, we see that $m = j$ is even. If $j > 2$, then we obtain $\Delta z_n > 0,\quad n \geq n_0$. Proceeding as in the proof of Theorem 2.7, we obtain a contradiction. If $j = 0$, then from Theorem 2.1 we have $\Delta z_n < 0,\quad n \geq n_0$. Thus, $z_n \to \ell$ as $n \to \infty$, where $0 < \ell < \infty$. For $\varepsilon \in (0, \ell)$, there exists an integer $n_1 > n_0$ such that

$$z_n = x_n + x_{n-\tau} > \ell - \varepsilon > 0, \quad n \geq n_1.$$

Hence, $\liminf_{n\to\infty} x_n > 0$. Proceeding as in the proof of Theorem 2.7, we obtain $\sum_{\ell=1}^{\infty}\sum_{i=1}^{r}\alpha_i(\ell) < \infty$, which is a contradiction to $\sum_{\ell=1}^{\infty}\sum_{i=1}^{r}\alpha_i(\ell) = \infty$. The proof is complete.

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