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Multi-Dimensional Integral Transform with Fox Function in Kernel in Lebesgue-Type Spaces

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Abstract: This paper is devoted to the study of the multi-dimensional integral transform with the Fox H-function in the kernel in weighted spaces with integrable functions in the domain $\mathbb{R}^n_+$ with positive coordinates. Due to the generality of the Fox H-function, many special integral transforms have the form studied in this paper, including operators with such kernels as generalized hypergeometric functions, classical hypergeometric functions, Bessel and modified Bessel functions and so on. Moreover, most important fractional integral operators, such as the Riemann–Liouville type, are covered by the class under consideration. The mapping properties in Lebesgue-weighted spaces, such as the boundedness, the range and the representations of the considered transformation, are established. In special cases, it is applied to the specific integral transforms mentioned above. We use a modern technique based on the extensive use of the Mellin transform and its properties. Moreover, we generalize our own previous results from the one-dimensional case to the multi-dimensional one. The multi-dimensional case is more complex and needs more delicate techniques.

Keywords: multi-dimensional integral transform; Fox H-function; Melling transform; weighted space; fractional integrals and derivatives

MSC: 44A30; 33C60; 35A22

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1. Introduction

We consider the multi-dimensional H-integral transform ([1], Formula (43)):

$$(\mathrm{H}f)(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\infty} \mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}} \left[\mathbf{xt} \left| \begin{array}{l} (\mathbf{a}_i, \overline{\alpha}_i)_{1,p} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,q} \end{array} \right. \right] f(\mathbf{t})\mathbf{dt}, \qquad \mathbf{x} > 0; \tag{1}$$

where (see [1,2], ch. 28; [3], ch. 1) $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$; $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$, $\mathbb{R}^n$ is the n-dimensional Euclidean space; $\mathbf{x} \cdot \mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for $\mathbf{1} = (1, 1, \ldots, 1)$. The inequality $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, \ldots, x_n > t_n$, and inequalities $\geq, <, \leq$ have similar meanings; $\int\limits_{\mathbf{0}}^{\infty} = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \cdots \int\limits_{0}^{\infty}$; by $\mathbb{N} = \{1, 2, \ldots\}$, we denote the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\mathbb{N}^n_0 = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$; $k = (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n_0$ $(k_i \in \mathbb{N}_0,\ i = 1, 2, \ldots, n)$ is a multi-index with $k! = k_1! \cdots k_n!$ and $|k| = k_1 + \cdots + k_n$; $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} > 0\}$; for $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n_+$ $\mathbf{D}^\kappa = \frac{\partial^{|\kappa|}}{(\partial x_1)^{\kappa_1} \cdots (\partial x_n)^{\kappa_n}}$; $\mathbf{dt} = dt_1 \cdots dt_n$; $\mathbf{t}^\kappa = t^{\kappa_1} t^{\kappa_2} \cdots t^{\kappa_n}$; $f(\mathbf{t}) = f(t_1, t_2, \ldots, t_n)$; $\mathbb{C}^n$ $(n \in \mathbb{N})$ is the n-dimensional space of n complex numbers $z = (z_1, z_2, \cdots, z_n)$ $(z_j \in \mathbb{C}, j = 1, 2, \cdots, n)$; $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$; $\overline{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n_+$; $\frac{\mathrm{d}}{\mathbf{dx}} = \frac{d}{dx_1 \cdot dx_2 \cdots dx_n}$;

$\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_0^n$ and $m_1 = m_2 = \cdots = m_n$; $\mathbf{n} = (\overline{n}_1, \overline{n}_2, \ldots, \overline{n}_n) \in \mathbb{N}_0^n$ and $\overline{n}_1 = \overline{n}_2 = \cdots = \overline{n}_n$; $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n$ and $p_1 = p_2 = \cdots = p_n$; $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{N}_0^n$ and $q_1 = q_2 = \cdots = q_n$ $(0 \le \mathbf{m} \le \mathbf{q}, 0 \le \mathbf{n} \le \mathbf{p})$;

$\mathbf{a}_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$, $1 \le i \le \mathbf{p}$, $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in \mathbb{C}$ $(i_1 = 1, 2, \ldots, p_1; \ldots; i_n = 1, 2, \ldots, p_n)$;

$\mathbf{b}_j = (b_{j_1}, b_{j_2}, \ldots, b_{j_n})$, $1 \le j \le \mathbf{q}$, $b_{j_1}, b_{j_2}, \ldots, b_{j_n} \in \mathbb{C}$ $(j_1 = 1, 2, \ldots, q_1; \ldots; j_n = 1, 2, \ldots, q_n)$;

$\overline{\alpha}_i = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n})$, $1 \le i \le \mathbf{p}$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n} \in \mathbb{R}_1^+$ $(i_1 = 1, 2, \ldots, p_1; \ldots; i_n = 1, 2, \ldots, p_n)$;

$\overline{\beta}_j = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n})$, $1 \le j \le \mathbf{q}$, $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n} \in \mathbb{R}_1^+$ $(j_1 = 1, 2, \ldots, q_1; \ldots; j_n = 1, 2, \ldots, q_n)$.

The function in the kernel of (1)

$$\mathrm{H}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}\left[\mathbf{xt}\,\middle|\, \begin{matrix} (\mathbf{a}_i, \overline{\alpha}_i)_{1,p} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,q} \end{matrix}\right] = \prod_{k=1}^{n} H_{p_k,q_k}^{m_k,\overline{n}_k}\left[x_k t_k \,\middle|\, \begin{matrix} (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{matrix}\right] \tag{2}$$

is the product of H-functions $H_{p,q}^{m,n}[z]$:

$$H_{p,q}^{m,n}[z] \equiv H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_L \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds, \; z \neq 0, \tag{3}$$

where

$$\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| s\right] = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}. \tag{4}$$

In the representation (3), L is a specifically chosen infinite contour, and the empty products, if any, are taken to be one.

The H-function (3) is the most general of the known special functions and includes, as special cases, elementary functions and special functions of the hypergeometric and Bessel type, as well as the Meyer G-function. One may find its properties, for example, in the books by Mathai and Saxena ([4], Ch. 2); Srivastava, Gupta and Goyal ([5], ch. 1); Prudnikov, Brychkov and Marichev ([6], Section 8.3); Kiryakova [7]; and Kilbas and Saigo ([8], Ch.1–Ch.4).

Due to the generality of the Fox H-function, many special integral transforms have the form studied in this paper, including operators with such kernels as generalized hypergeometric functions, classical hypergeometric functions, Bessel and modified Bessel functions and so on. Moreover, most important fractional integral operators, such as the Riemann–Liouville type, are covered by the class under consideration. The mapping properties in Lebesgue-weighted spaces, such as the boundedness, the range and the representations of the considered transformation, are established. In special cases, it is applied to the specific integral transforms mentioned above. We use a modern technique based on the extensive use of the Mellin transform and its properties.

Our paper is devoted to the study of the H-transform (1) in Lebesgue-type weighted spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ of functions $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ on $\mathbb{R}_+^{\mathbf{n}}$, such that

$$\|f\|_{\overline{\nu},\overline{2}} = \{\int_{\mathbb{R}_+^1} x_n^{2\cdot\nu_n - 1}\{\cdots\{\int_{\mathbb{R}_+^1} x_2^{2\cdot\nu_2\cdot - 1}\times$$

$$[\int_{\mathbb{R}_+^1} x_1^{2\cdot\nu_1 - 1}|f(x_1, \ldots, x_n)|^2 dx_1]dx_2\}\cdots\}dx_n\}^{1/2} < \infty,$$

$\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = \cdots = \nu_n$, and $\overline{2} = (2, 2, \ldots, 2)$.

In this paper, we apply some our previous results to obtain mapping properties such as the boundedness, the range and the representations for the H-transform (1).

The research results for transformation (1) generalize those obtained earlier for the corresponding one-dimensional transformation (see [8], Ch. 3):

$$(\mathrm{H}f)(x) = \int_{\mathbf{0}}^{\infty} \mathrm{H}_{p,q}^{m,n}\left[xt\middle| \begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array}\right] f(t)dt, \qquad x>0; \tag{5}$$

in the space $\mathfrak{L}_{\nu,2}$ of Lebesgue measurable functions f on $\mathbb{R}^1_+ = (0,\infty)$, such that

$$\int_0^\infty |t^\nu f(t)|^2 \frac{dt}{t} < \infty \quad (\nu \in \mathbb{R}).$$

The H-transform (5) generalizes many integral transforms: transforms with the Meijer G-function, Laplace and Hankel transforms, transforms with Gauss hypergeometric functions and transforms with other hypergeometric and Bessel functions in the kernels. One may find a survey of results and a bibliography in this field for the one-dimensional case in a monograph ([8], Sections 6–8). Note that a very important class of transforms under consideration is the class of Buschman–Erdélyi operators; they have many important properties and applications. The topic of this paper is also strongly connected with transmutation theory, cf. [9].

Note that, in transmutation theory applied to differential equations, its solutions are represented as integral transforms; in this way, solutions of perturbed differential equations are represented via more simple solutions of unperturbed equations. Through the results of this paper and similar ones, such a representation may also be accompanied by norm estimates in classical functional spaces. It helps to estimate the norms of perturbed equations and analyze their smoothness or singularity conditions, cf. [9].

2. Preliminaries

The properties of the H-function $H_{p,q}^{m,n}[z]$ (3) depend on the following numbers ([8], Formulas 1.1.7–1.1.15):

$$a^* = \sum_{i=1}^{n}\alpha_i - \sum_{i=n+1}^{p}\alpha_i + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j; \;\; \Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i; \tag{6}$$

$$\delta = \prod_{i=1}^{p}\alpha_i^{-\alpha_i}\prod_{j=1}^{q}\beta_j^{\beta_j}; \tag{7}$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}; \tag{8}$$

$$a_1^* = \sum_{j=1}^{m}\beta_j - \sum_{i=n+1}^{p}\alpha_i; \; a_2^* = \sum_{i=1}^{n}\alpha_i - \sum_{j=m+1}^{q}\beta_j; \; a_1^* + a_2^* = a^*, \;\; a_1^* - a_2^* = \Delta; \tag{9}$$

$$\xi = \sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j + \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i; \tag{10}$$

$$c^* = m+n-\frac{p+q}{2}. \tag{11}$$

The empty sum in (6), (8), (9), (10) and the empty product in (7), if they occur, are taken to be zero and one, respectively.

The following assertions hold.

Lemma 1 ([8], Lemma 1.2). *For σ, $t \in \mathbb{R}$, the following estimate holds*

$$|\mathcal{H}_{p,q}^{m,n}(\sigma+it)| \sim C|t|^{\Delta\sigma+\mathrm{Re}(\mu)} \exp^{-\pi[|t|a^*+\mathrm{Im}(\xi)\mathrm{sign}(t)]/2} \quad (|t| \to \infty) \tag{12}$$

uniformly in σ on any bounded interval in $\mathbb{R}$, where

$$C = (2\pi)^{c^*} \exp^{-c^*-\Delta\sigma-\mathrm{Re}(\mu)} \delta^{\sigma} \prod_{i=1}^{p} \alpha_i^{1/2-\mathrm{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\mathrm{Re}(b_j)-1/2} \tag{13}$$

and ξ and c^* are defined in (10) and (11).

Theorem 1 ([8],Theorem 3.4). *Let $\alpha < \zeta < \beta$ and either of the conditions $a^* > 0$ or $a^* = 0$ and $\Delta\zeta + \mathrm{Re}(\mu) < -1$ hold. Then, for $x > 0$, except for $x = \delta$ when $a^* = 0$ and $\Delta = 0$, the relation*

$$H_{p,q}^{m,n}\left[x \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix}\right] = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix} \middle| t\right] x^{-t} dt \tag{14}$$

holds and the estimate

$$|H_{p,q}^{m,n}\left[x \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_p, \beta_p) \end{matrix}\right]| \leq A_\zeta x^{-\zeta} \tag{15}$$

is valid, where A_ζ is a positive constant depending only on ζ.

A set of bounded linear operators acting from a Banach space X into a Banach space Y is denoted by $[X, Y]$.

The multi-dimensional Mellin integral transform $(\mathfrak{M}f)(\mathbf{x})$ of function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$, is determined by the formula

$$(\mathfrak{M}f)(\mathbf{s}) = \int_0^\infty f(\mathbf{t})\mathbf{t}^{\mathbf{s}-1}\mathbf{dt}, \ \mathrm{Re}(\mathbf{s}) = \overline{\nu}, \tag{16}$$

$\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$. The inverse multi-dimensional Mellin transform has the form

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \frac{1}{(2\pi i)^n} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \cdots \int_{\gamma_n-i\infty}^{\gamma_n+i\infty} \mathbf{x}^{-\mathbf{s}} g(\mathbf{s}) \mathrm{d}\mathbf{s}, \tag{17}$$

$\mathbf{x} \in \mathbb{R}_+^n$, $\gamma_j = \mathrm{Re}(s_j)$ $(j = 1, \cdots, n)$. The theory of multi-dimensional integral transformations (16) and (17) can be recognized, for example, in books ([3], Ch. 1; [10,11]).

We will need the following spaces. As usual, by $\mathrm{L}_{\overline{p}}(\mathbb{R}^n)$, we understand the space of functions $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, for which

$$\|f\|_{\overline{p}} = \left\{ \int_{\mathbb{R}^n} |f(\mathbf{x})|^{\overline{p}} \mathrm{d}\mathbf{x} \right\}^{1/\overline{p}} < \infty, \ \overline{p} = (p_1, p_2, \ldots, p_n), \ 1 \leq \overline{p} < \infty.$$

If $\overline{p} = \infty$, then the space $\mathrm{L}_\infty(\mathbb{R}^n)$ is defined as the collection of all measurable functions with a finite norm

$$\|f\|_{\mathrm{L}_\infty(\mathbb{R}^n)} = esssup|f(\mathbf{x})|,$$

where $esssup\,|f(\mathbf{x})|$ is the essential supremum of the function $|f(\mathbf{x})|$ [12].

We need the following properties of the Mellin transform (16).

Lemma 2 ([1], Lemma 1). *Let $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = \cdots = \nu_n$. The following properties of the Mellin transform (16) are valid.*

(a) Transformation (16) is a unitary mapping of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto the space $\mathrm{L}_{\overline{2}}(\mathbb{R}^n)$.

(b) For $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, the following holds

$$f(\mathbf{x}) = \frac{1}{(2\pi i)^n} \lim_{R\to\infty} \int_{\nu_1 - iR}^{\nu_1 + iR} \int_{\nu_2 - iR}^{\nu_2 + iR} \cdots \int_{\nu_n - iR}^{\nu_n + iR} (\mathfrak{M}f)(\mathbf{s})\mathbf{x}^{-\mathbf{s}} d\mathbf{s}, \tag{18}$$

where the limit is taken in the topology of the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ and where

if $F(\overline{\nu} + i\mathbf{t}) = \prod_{i=1}^{n} F_j(\nu_j + it_j)$, $F_j(\nu_j + it_j) \in L_1(-R, R)$, $j = 1, 2, \ldots, n$, then

$$\int_{\nu_1 - iR}^{\nu_1 + iR} \int_{\nu_2 - iR}^{\nu_2 + iR} \cdots \int_{\nu_n - iR}^{\nu_n + iR} F(\mathbf{s}) d\mathbf{s} = i^n \int_{-R}^{R} \int_{-R}^{R} \cdots \int_{-R}^{R} F(\overline{\nu} + i\mathbf{t}) d\mathbf{t}.$$

(c) For functions $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$, the following equality holds

$$\int_{\mathbf{0}}^{\infty} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} = \frac{1}{(2\pi i)^n} \int_{\overline{\nu} - i\infty}^{\overline{\nu} + i\infty} (\mathfrak{M}f)(\mathbf{s})(\mathfrak{M}g)(1 - \mathbf{s})\mathbf{x}^{-\mathbf{s}} d\mathbf{s}. \tag{19}$$

In [1], we consider the general multi-dimensional integral transform ([1], Formula (1))

$$(\mathrm{K}f)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{d}{d\mathbf{x}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \int_{\mathbf{0}}^{\infty} \mathrm{k}[\mathbf{x}\mathbf{t}] f(\mathbf{t}) d\mathbf{t} \quad (\mathbf{x} > 0), \tag{20}$$

where the function $\mathrm{k}[\mathbf{x}\mathbf{t}]$ in the kernel of (20) is the product of one type of special function:

$$\mathrm{k}[\mathbf{x}\mathbf{t}] = \mathrm{k}[x_1 t_1] \cdot \mathrm{k}[x_2 t_2] \cdots \mathrm{k}[x_n t_n].$$

Transformation (20) satisfies the following theorem.

Theorem 2 ([1], Theorem 1). *Let $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = \cdots = \nu_n)$, $\overline{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{R}_+^n$, and $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$.*

(a) If the transformation operator (20) satisfies the condition $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$, then the kernel on the right side of (20) $\mathrm{k} \in \mathfrak{L}_{1-\overline{\nu},\overline{2}}$. If we set, for $\nu_j \neq 1 - (\mathrm{Re}(\lambda_j) + 1)/h_j$, $j = 1, 2, \ldots, n$,

$$\begin{aligned}(\mathfrak{M}\mathrm{k})(1 - \overline{\nu} + i\mathbf{t}) &= \frac{\theta(\mathbf{t})}{\overline{\lambda} + 1 - (1 - \overline{\nu} + i\mathbf{t})\overline{h}} \\ &= \prod_{j=1}^{n} \frac{\theta(t_j)}{\lambda_j + 1 - (1 - \nu_j + it_j)h_j}\end{aligned} \tag{21}$$

almost everywhere, then function $\theta \in L_\infty(\mathbb{R}^n)$, and, for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, the relation

$$(\mathfrak{M}\mathrm{K}f)(1 - \overline{\nu} + i\mathbf{t}) = \theta(\mathbf{t})(\mathfrak{M}f)(\overline{\nu} - i\mathbf{t}) \tag{22}$$

holds almost everywhere.

(b) Conversely, for a given function $\theta \in L_\infty(\mathbb{R}^n)$, there is a transform $\mathrm{K} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ so that the equality (22) holds for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. Moreover, if $\nu_j \neq 1 - (\mathrm{Re}(\lambda_j) + 1)/h_j$, $j = 1, 2, \ldots, n$, then transformation $\mathrm{K}f$ (20) is representable in the form (20) with the kernel k defined by (21).

(c) Based on statement (a) or (b) with $\theta \neq 0$, K is a one-to-one transformation from the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ into the space $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$, and if, in addition, $1/\theta \in L_\infty(\mathbb{R}^n)$, then K maps $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$, and, for functions f, $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, the relation

$$\int_{\mathbf{0}}^{\infty} f(\mathbf{x})(\mathrm{K}g)(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{0}}^{\infty} (\mathrm{K}f)(\mathbf{x}) g(\mathbf{x}) d\mathbf{x} \tag{23}$$

is valid.

3. $\mathfrak{L}_{\overline{\nu},2}$-Theory for the Multi-Dimensional H-Transform

To formulate the results for the transform Hf (1), we need the following constants ([1]), which are analogous for the one-dimensional case defined via the parameters of the H-function (3) ([8], (3.4.1), (3.4.2), (1.1.7), (1.1.8), (1.1.10)).

Let $\widetilde{\alpha} = (\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n)$ and $\widetilde{\beta} = (\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n)$, where

$$\widetilde{\alpha}_1 = \begin{cases} -\min\limits_{1 \le j_1 \le m_1} \left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right], & m_1 > 0, \\ -\infty, & m_1 = 0, \end{cases} \quad \widetilde{\beta}_1 = \begin{cases} \min\limits_{1 \le i_1 \le \overline{n}_1} \left[\frac{1-\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & \overline{n}_1 > 0, \\ \infty, & \overline{n}_1 = 0, \end{cases}$$

$$\widetilde{\alpha}_2 = \begin{cases} -\min\limits_{1 \le j_2 \le m_2} \left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right], & m_2 > 0, \\ -\infty, & m_2 = 0, \end{cases} \quad \widetilde{\beta}_2 = \begin{cases} \min\limits_{1 \le i_2 \le \overline{n}_2} \left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & \overline{n}_2 > 0, \\ \infty, & \overline{n}_2 = 0, \end{cases}$$

and

$$\widetilde{\alpha}_n = \begin{cases} -\min\limits_{1 \le j_n \le m_n} \left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right], & m_n > 0, \\ -\infty, & m_2 = 0, \end{cases} \quad \widetilde{\beta}_n = \begin{cases} \min\limits_{1 \le i_n \le \overline{n}_n} \left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & \overline{n}_n > 0, \\ \infty, & \overline{n}_n = 0; \end{cases} \tag{24}$$

and let $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$, $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ and

$$a_1^* = \sum_{i=1}^{\overline{n}_1} \alpha_{i_1} - \sum_{i=\overline{n}_1+1}^{p_1} \alpha_{i_1} + \sum_{j=1}^{m_1} \beta_{j_1} - \sum_{j=m_1+1}^{q_1} \beta_{j_1}, \ \Delta_1 = \sum_{j=1}^{q_1} \beta_{j_1} - \sum_{i=1}^{p_1} \alpha_{i_1},$$

$$a_2^* = \sum_{i=1}^{\overline{n}_2} \alpha_{i_2} - \sum_{i=\overline{n}_2+1}^{p_2} \alpha_{i_2} + \sum_{j=1}^{m_2} \beta_{j_2} - \sum_{j=m_1+1}^{q_2} \beta_{j_2}, \ \Delta_2 = \sum_{j=1}^{q_2} \beta_{j_2} - \sum_{i=1}^{p_2} \alpha_{i_2},$$

and

$$a_n^* = \sum_{i=1}^{\overline{n}_n} \alpha_{i_n} - \sum_{i=\overline{n}_n+1}^{p_n} \alpha_{i_n} + \sum_{j=1}^{m_n} \beta_{j_n} - \sum_{j=m_n+1}^{q_n} \beta_{j_n}; \Delta_n = \sum_{j=1}^{q_n} \beta_{j_n} - \sum_{i=1}^{p_n} \alpha_{i_n}; \tag{25}$$

and let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and

$$\mu_1 = \sum_{j=1}^{q_1} b_{j_1} - \sum_{i=1}^{p_1} a_{i_1} + \frac{p_1 - q_1}{2}, \mu_2 = \sum_{j=1}^{q_2} b_{j_2} - \sum_{i=1}^{p_2} a_{i_2} + \frac{p_2 - q_2}{2}, \ldots,$$

$$\mu_n = \sum_{j=1}^{q_n} b_{j_n} - \sum_{i=1}^{p_n} a_{i_n} + \frac{p_n - q_n}{2}; \tag{26}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}(\mathbf{s})$

$$\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}(\mathbf{s}) \equiv \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}} \left[\begin{matrix} (\mathbf{a}_i, \overline{\alpha}_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \overline{\beta}_j)_{1,\mathbf{q}} \end{matrix} \middle| \mathbf{s} \right] = \prod_{k=1}^{n} \mathcal{H}^{m_k, \overline{n}_k}_{p_k, q_k} \left[\begin{matrix} (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{matrix} \middle| s \right], \tag{27}$$

is called a set of vectors $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = \cdots = \nu_n)$, such that $\widetilde{\alpha}_k < 1 - \nu_k < \widetilde{\beta}_k$, $k = 1, 2, \ldots n$, where the parameters $\widetilde{\alpha}_k$, $\widetilde{\beta}_k (k = 1, 2, \ldots, n)$ are defined by Formula (24), and functions $\mathcal{H}^{m_k, \overline{n}_k}_{p_k, q_k}(s_k)$ $(k = 1, 2, \ldots, n)$ of the view (4) have zeros on lines $\mathrm{Re}(s_k) < 1 - \nu_k$ $(k = 1, 2, \ldots, n)$, respectively.

Applying the multi-dimensional Mellin transformation (16) to (1), formally, we obtain

$$(\mathfrak{M}\mathrm{H}f)(\mathbf{s}) = \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}} \left[\begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix} |\mathbf{s} \right] (\mathfrak{M}f)(1 - \mathbf{s}). \tag{28}$$

Theorem 3. *Suppose that*

$$\widetilde{\alpha}_k < 1-\nu_k < \widetilde{\beta}_k;\ \nu_k = \nu_l,\ k \neq l\ (k,l = 1,2,\ldots,n); \tag{29}$$

and that either of the conditions

$$a_k^* > 0\ (k = 1,2,\ldots,n); \tag{30}$$

or

$$a_k^* = 0, \Delta_k[1-\nu_k] + \mathrm{Re}(\mu_k) \leq 0\ (k = 1,2,\ldots,n) \tag{31}$$

holds. Then, we have the following results.

(a) There exists a one-to-one transform $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ *so that the relation (28) holds for* $\mathrm{Re}(\mathbf{s}) = 1-\overline{\nu}$ *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$.

If $a_k^* = 0, \Delta_k[1-\nu_k] + \mathrm{Re}(\mu_k) = 0\ (k = 1,2,\ldots,n)$, *and* $\overline{\nu}$ *does not belong to an exceptional set* $\mathcal{E}_{\overline{\mathcal{H}}}$, *then the operator* H *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$.

(b) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then, for H, we have the relation (23)*

$$\int_0^\infty f(\mathbf{x})(\mathrm{H}g)(\mathbf{x})\mathbf{dx} = \int_0^\infty (\mathrm{H}f)(\mathbf{x})g(\mathbf{x})d\mathbf{x}. \tag{32}$$

(c) Let $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, $\overline{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_n) \in \mathbb{C}^n$, $\overline{h} = (h_1,h_2,\ldots,h_n) \in \mathbb{R}_+^n$. *If* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$, *then* $\mathrm{H}f$ *is given by the formula*

$$(\mathrm{H}f)(\mathbf{x}) = \overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$

$$\times\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\left|\begin{array}{c}(-\overline{\lambda},\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h})\end{array}\right.\right] f(\mathbf{t})\mathbf{dt}. \tag{33}$$

When $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, $\mathrm{H}f$ *is given by*

$$(\mathrm{H}f)(\mathbf{x}) = -\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}}$$

$$\times\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}}\int_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\left|\begin{array}{c}(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}},(-\overline{\lambda},\overline{h})\\ (-\overline{\lambda}-1,\overline{h}),(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}}\end{array}\right.\right] f(\mathbf{t})\mathbf{dt}. \tag{34}$$

(d) The transform H *is independent of* $\overline{\nu}$ *in the sense that, for* $\overline{\nu}$ *and* $\widetilde{\overline{\nu}}$ *satisfying the assumptions (29), and either (30) or (31), and for the respective transforms* H *on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\widetilde{\mathrm{H}}$ *on* $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ *given in (28), then* $\mathrm{H}f = \widetilde{\mathrm{H}}f$ *for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}} \bigcap \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$.

Proof. Let $\overline{\omega}(\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{\nu}+i\mathbf{t}) = \prod_{k=1}^{n}\mathcal{H}(1-\nu_k+it_k)$. By virtue of (4), (24), and the conditions (29), the functions $\mathcal{H}^{m_1,\overline{n}_1}_{p_1,q_1}(s_1), \mathcal{H}^{m_2,\overline{n}_2}_{p_2,q_2}(s_2),\ldots,\mathcal{H}^{m_n,\overline{n}_n}_{p_n,q_n}(s_n)$ are analytic in the strips $\widetilde{\alpha}_1 < 1-\nu_1 < \widetilde{\beta}_1,\ldots,\widetilde{\alpha}_n < 1-\nu_n < \widetilde{\beta}_n, \nu_1 = \nu_2 = \cdots = \nu_n$, respectively. In accordance with (12) and conditions (30) or (31), $\overline{\omega}(\mathbf{t}) = O(1)$ as $|\mathbf{t}| \to \infty$. Therefore, $\overline{\omega} \in \mathrm{L}_\infty(\mathbb{R}^n)$, and hence we obtain from Theorem 2 (b) that there exists a transform $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ such that

$$(\mathfrak{M}\mathrm{H}f)(\mathbf{s})(1-\overline{\nu}+i\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{\nu}+i\mathbf{t})(\mathfrak{M}f)(\overline{\nu}-i\mathbf{t})$$

for $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. This means that the equality (28) holds when condition $\mathrm{Re}(\mathbf{s}) = 1-\overline{\nu}$ is met. Since the functions $\mathcal{H}^{m_1,\overline{n}_1}_{p_1,q_1}(s_1), \mathcal{H}^{m_2,\overline{n}_2}_{p_2,q_2}(s_2),\ldots,\mathcal{H}^{m_n,\overline{n}_n}_{p_n,q_n}(s_n)$ are analytic in the strips $\widetilde{\alpha}_1 < 1-\nu_1 < \widetilde{\beta}_1,\ldots,\widetilde{\alpha}_n < 1-\nu_n < \widetilde{\beta}_n, \nu_1 = \nu_2 = \cdots = \nu_n$, respectively, and have isolated zeros, then $\overline{\omega}(\mathbf{t}) \neq 0$ almost everywhere. Thus, it follows from Theorem 2 (c) that $\mathrm{H} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{1-\overline{\nu},\overline{2}}]$ is a one-to-one transform. If $a_k^* = 0$, $\Delta_k(1-\nu_k) + \mathrm{Re}(\mu_k) = 0$

$(k = 1, 2, \ldots n)$ and $\overline{\nu}$ is not in the exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of $\overline{\mathcal{H}}$, then $1/\overline{\omega} \in L_\infty(\mathbb{R}^n)$, and, from Theorem 2 (c), we have that H transforms the space $\mathfrak{L}_{\overline{\nu},\overline{2}}$ onto $\mathfrak{L}_{1-\overline{\nu},\overline{2}}$. This completes the proof of the statement (a) of the theorem.

According to the statement of the Theorem 2 (c), if $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, then the relation (32) is valid. Thus, the assertion (b) is true.

Let us prove the validity of the representation (33). Suppose that $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$. To show the relation (33), it is sufficient to calculate the kernel k in the transform (20) for such $\overline{\lambda}$. From (21), we obtain the equality

$$(\mathfrak{M}\mathrm{k})(1-\overline{\nu}+i\mathbf{t}) = \overline{\mathcal{H}}(1-\overline{\nu}+i\mathbf{t})\frac{1}{\overline{\lambda}+1-(1-\overline{\nu}+i\mathbf{t})\overline{h}}$$
$$= \prod_{k=1}^{n} \mathcal{H}(1-\nu_k+it_k)\frac{1}{\lambda_k+1-(1-\nu_k+it_k)h_k}$$

or, for $\mathrm{Re}(\mathbf{s}) = 1-\overline{\nu}$,

$$(\mathfrak{M}\mathrm{k})(\mathbf{s}) = \overline{\mathcal{H}}(\mathbf{s})\frac{1}{\overline{\lambda}+1-\overline{h}\mathbf{s}} = \prod_{k=1}^{n} \mathcal{H}(s_k)\frac{1}{\lambda_k+1-h_k s_k}. \tag{35}$$

Then, from (18) and (35), we obtain the expression for the kernel k

$$\mathrm{k}(\mathbf{x}) = \prod_{k=1}^{n} \mathrm{k}(x_k) = \frac{1}{(2\pi i)^n}\prod_{k=1}^{n} \lim_{R\to\infty}\int_{1-\nu_k-iR}^{1-\nu_k+iR} (\mathfrak{M}\mathrm{k})(s_k) x_k^{-s_k} ds_k$$
$$= \frac{1}{(2\pi i)^n}\prod_{k=1}^{n} \lim_{R\to\infty}\int_{1-\nu_k-iR}^{1-\nu_k+iR} \mathcal{H}_k(s_k)\frac{1}{\lambda_k+1-h_k s_k} x_k^{-s_k} ds_k, \tag{36}$$

where the limits are taken in the topology of $\mathfrak{L}_{\nu,2}$.

According to (4) and (27), we have

$$\overline{\mathcal{H}}(\mathbf{s})\frac{1}{\overline{\lambda}+1-\overline{h}\mathbf{s}} = \overline{\mathcal{H}}(\mathbf{s})\frac{\Gamma(1-(-\overline{\lambda})-\overline{h}\mathbf{s})}{\Gamma(1-(-\overline{\lambda}-1)-\overline{h}\mathbf{s})}$$
$$= \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\,\mathbf{q}+1}\left[\begin{matrix}(-\overline{\lambda},\,\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\,(-\overline{\lambda}-1,\,\overline{h})\end{matrix}\Big|\mathbf{s}\right]$$
$$= \prod_{k=1}^{n} \mathcal{H}^{m_k,\overline{n}_k+1}_{p_k+1,\,q_k+1}\left[\begin{matrix}(-\lambda_k,\,h_k),(a_{i_k},\alpha_{i_k})_{1,p_k}\\ (b_{j_k},\beta_{j_k})_{1,q_k},\,(-\lambda_k-1,\,h_k)\end{matrix}\Big|s_k\right]. \tag{37}$$

Denote by $\hat{\alpha}_k, \hat{\beta}_k$ $(k = 1, 2, \ldots, n)$ the constants $\widetilde{\alpha}_k, \widetilde{\beta}_k$ $(k = 1, 2, \ldots, n)$ in (24), respectively; by $\widetilde{a}^*_k$ $(k = 1, 2, \ldots, n)$, the constants a^*_k $(k = 1, 2, \ldots, n)$; and by $\widetilde{\Delta}_k$ $(k = 1, 2, \ldots, n)$, the constants Δ_k $(k = 1, 2, \ldots, n)$ in (25), respectively; and by $\widetilde{\mu}_k$ $(k = 1, 2, \ldots, n)$, the constants μ_k $(k = 1, 2, \ldots, n)$ in (26), respectively, for $\mathcal{H}^{m_k,\overline{n}_k+1}_{p_k+1,q_k+1}$ $(k = 1, 2, \ldots, n)$ in (37). Then, $\hat{\alpha}_k = \widetilde{\alpha}_k$ $(k = 1, 2, \ldots, n)$; $\hat{\beta}_k = \min[\widetilde{\beta}_k, (1+\mathrm{Re}(\lambda_k))/h_k]$ $(k = 1, 2, \ldots, n)$; $\widetilde{a}^*_k = a^*_k$ $(k = 1, 2, \ldots, n)$; $\widetilde{\Delta}_k = \Delta_k$ $(k = 1, 2, \ldots, n)$; $\widetilde{\mu}_k = \mu_k - 1$ $(k = 1, 2, \ldots, n)$. Thus, it follows that

(a') $\hat{\alpha}_k < 1-\nu_k < \hat{\beta}_i$ $(k = 1, 2, \ldots, n)$;
from $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$, and either of the conditions
(b') $\widetilde{a}^*_k > 0$ $(k = 1, 2, \ldots, n)$;
(c') $\widetilde{a}^*_k = 0$ $(k = 1, 2, \ldots, n)$; or
$\widetilde{\Delta}_k(1-\nu_k) + \mathrm{Re}(\widetilde{\mu}_k) = \Delta_k(1-\nu_k) + \mathrm{Re}(\mu_k) - 1 \le -1$
$(k = 1, 2, \ldots, n)$ holds. Applying Theorem 1 for $\mathbf{x} > 0$, then the equality

$$\mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{xt}\left|\begin{array}{c}(-\overline{\lambda},\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h})\end{array}\right.\right]$$
$$=\prod_{k=1}^{n}\mathrm{H}^{m_k,\overline{n}_k+1}_{p_k+1,q_k+1}\left[x_k\left|\begin{array}{c}(-\lambda_k,\ h_k),(a_{i_k},\alpha_{i_k})_{1,p_k}\\(b_{j_k},\beta_{j_k})_{1,q_k},\ (-\lambda_k-1,\ h_k)\end{array}\right.\right]$$
$$=\frac{1}{(2\pi i)^n}\prod_{k=1}^{n}\lim_{\mathrm{R}\to\infty}\int_{1-\nu_k-i\mathrm{R}}^{1-\nu_k+i\mathrm{R}}\mathcal{H}_k(s_k)\frac{1}{\lambda_k+1-h_ks_k}x_k^{-s_k}ds_k \tag{38}$$

holds almost everywhere. Then, (36) and (38) lead to the fact that the kernel k is given by

$$\mathrm{k}(\mathbf{x})=\mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\mathbf{x}\left|\begin{array}{c}(-\overline{\lambda},\overline{h}),(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}}\\(\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h})\end{array}\right.\right],$$

and (33) is proven.

The representation (34) is proven similarly to (33). We use the equality

$$\overline{\mathcal{H}}(\mathbf{s})\frac{1}{\overline{\lambda}+1-\overline{h}\mathbf{s}}=-\overline{\mathcal{H}}(\mathbf{s})\frac{\Gamma(\overline{h}\mathbf{s}-\overline{\lambda}-1)}{\Gamma(\overline{h}\mathbf{s}-\overline{\lambda})}$$
$$=-\overline{\mathcal{H}}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\,\mathbf{q}+1}\left[\begin{array}{c}(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}},(-\overline{\lambda},\ \overline{h})\\(-\overline{\lambda}-1,\ \overline{h}),\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}}\end{array}\middle|\mathbf{s}\right]$$
$$=-\prod_{k=1}^{n}\mathcal{H}^{m_k+1,\overline{n}_k}_{p_k+1,q_k+1}\left[\begin{array}{c}(a_{i_k},\alpha_{i_k})_{1,p_k},(-\lambda_k,\ h_k)\\(-\lambda_k-1,\ h_k),\ (b_{j_k},\beta_{j_k})_{1,q_k}\end{array}\middle|s_k\right]. \tag{39}$$

instead of (37). Thus, the statement (c) is proven. □

Let us prove (d). If $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}\bigcap\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ and $\mathrm{Re}(\overline{\lambda})>\max[(1-\overline{\nu})\overline{h}-1,(1-\widetilde{\overline{\nu}})\overline{h}-1]$ or $\mathrm{Re}(\overline{\lambda})<\min[(1-\overline{\nu})\overline{h}-1,(1-\widetilde{\overline{\nu}})\overline{h}-1]$, then both transforms $\mathrm{H}f$ and $\widetilde{\mathrm{H}}f$ are given in (33) or (34), respectively, which shows that they are independent of $\overline{\nu}$.

Corollary 1. *Suppose that* $\widetilde{\alpha}_k<\widetilde{\beta}_k$ $(k=1,2,\dots,n)$, *and that one of the following conditions holds:*
(a) $a_k^*>0$ $(k=1,2,\dots,n)$;
(b) $a_k^*=0$ $(k=1,2,\dots,n)$; $\Delta_k>0$ $(k=1,2,\dots,n)$; *and*
$\widetilde{\alpha}_k<-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$;
(c) $a_k^*=0$; $\Delta_k<0$ $(k=1,2,\dots,n)$; *and*
$\widetilde{\beta}_k>-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$;
(d) $a_k^*=0$ $(k=1,2,\dots,n)$; $\Delta_k=0$, $(k=1,2,\dots,n)$; *and*
$\mathrm{Re}(\mu_k)\le 0$ $(k=1,2,\dots,n)$.
Then the H*-transform (1) can be defined on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *with*
$\widetilde{\alpha}_k<\nu_k<\beta_k$ $(k=1,2,\dots,n)$; $\nu_1=\nu_2=\cdots=\nu_n$.

Proof. When $1-\widetilde{\beta}_k<\nu_k<1-\widetilde{\alpha}_k$ $(k=1,2,\dots,n)$, by Theorem 3, if either $a_k^*>0$ $(k=1,2,\dots,n)$ or $a_k^*=0$ $(k=1,2,\dots,n)$, $\Delta_k(1-\nu_k)\mathrm{Re}(\mu_k)\le 0$ $(k=1,2,\dots,n)$ is satisfied, then the H-transform can be defined on $\mathfrak{L}_{\overline{\nu},\overline{2}}$, which is also valid when $\widetilde{\alpha}_k<\nu_k<\widetilde{\beta}_k$ $(k=1,2,\dots,n)$. Hence, the corollary is clear in cases (a) and (d). When $\Delta_k>0$ and $\widetilde{\alpha}_k<-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$, the assumption $\widetilde{\alpha}_k<\widetilde{\beta}_k$ $(k=1,2,\dots,n)$ yields that there exists a vector $\overline{\nu}=(\nu_1,\nu_2,\dots,\nu_n)$ such that $\widetilde{\alpha}_k<1-\nu_k\le-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$, and $\alpha_k<1-\nu_k\le-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$, which are required. For the case (c), the situation is similar, i.e., there exists $\overline{\nu}$ of the forms $\widetilde{\beta}_k>1-\nu_k\ge-\frac{\mathrm{Re}(\mu_k)}{\Delta_k}$ $(k=1,2,\dots,n)$ and $\widetilde{\alpha}_k<1-\nu_k$ $(k=1,2,\dots,n)$. Thus, the proof is completed. □

4. Conclusions

The multi-dimensional integral transformation with the Fox H-function is studied. Conditions are obtained for the boundedness and one-to-oneness of the operator of such a transformation from one Lebesgue-type weighted space of functions to another, and the analogues of the formula for integration by parts are proven. For the transformation under consideration, various integral representations are established. The results generalize those obtained earlier for the corresponding one-dimensional integral transform.

Due to the generality of the Fox H-function, many special integral transforms have the form studied in this paper, including operators with such kernels as generalized hypergeometric functions, classical hypergeometric functions, Bessel and modified Bessel functions and so on. Moreover, most important fractional integral operators, such as the Riemann–Liouville type, are covered by the class under consideration. The mapping properties in Lebesgue-weighted spaces, such as the boundedness, the range and the representations of the considered transformation, are established. In special cases, it is applied to the specific integral transforms mentioned above. We use a modern technique based on the extensive use of the Mellin transform and its properties. Moreover, we generalize our own previous results from the one-dimensional case to the multi-dimensional one. The multi-dimensional case is more complex and needs more delicate techniques.

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